

Réécriture modulo dans les catégories diagrammatiques.

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Soutenance de thèse de Doctorat

Sous la direction de Philippe Malbos, Stéphane Gaussent et Alistair Savage

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Université Claude Bernard



Lyon 1



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I. Introduction

II. Convergent presentation of the Khovanov-Lauda-Rouquier algebras

III. Confluence modulo isotopies in the Khovanov-Lauda-Rouquier 2-category

IV. Conclusion and perspectives

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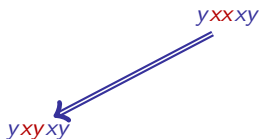
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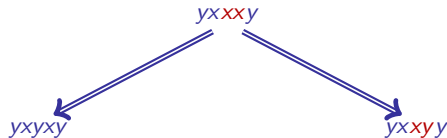
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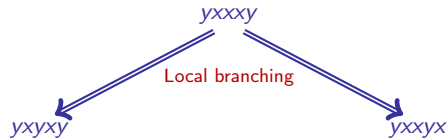
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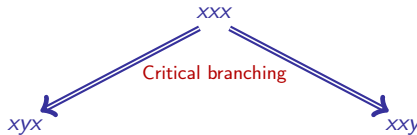
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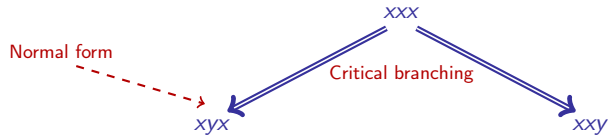
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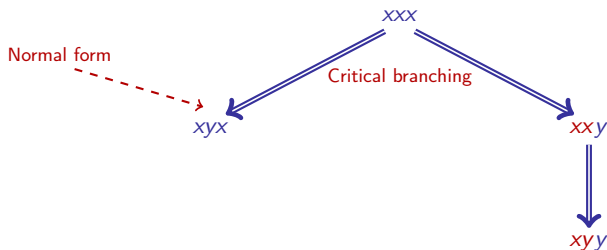
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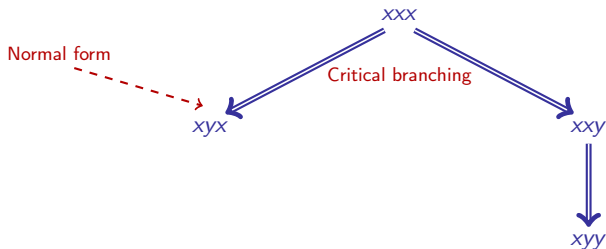


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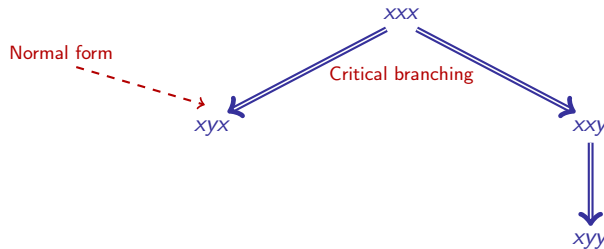


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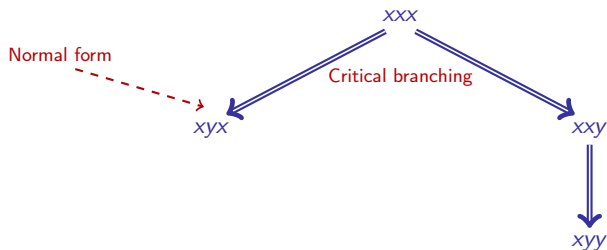
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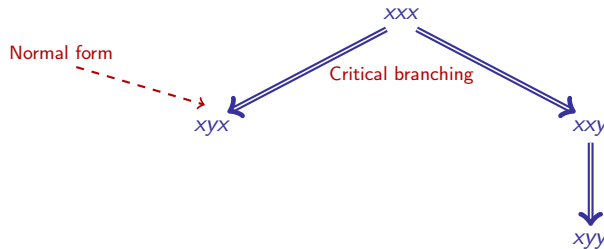
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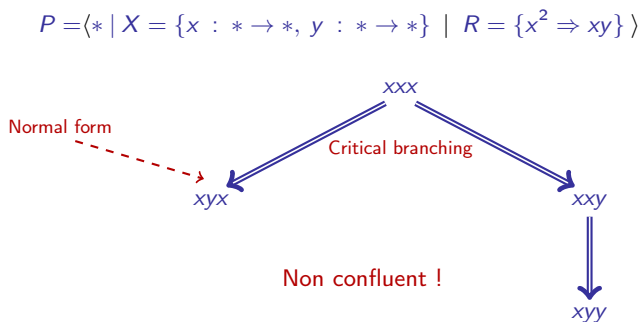


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- ii) The vector space $P_1^\ell := \mathbb{K}[P_1^*]$ admits the direct decomposition

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- ▶ Several new questions, e.g. extension to rewriting in 2-supercategories (with **M. Ebert** and **A. Lauda**) and explicit proofs of categorification (with **G. Naisse**).

II. Convergent presentation of the KLR algebras

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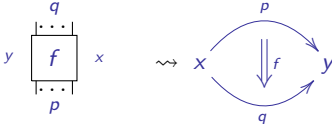
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► We study these algebras by realizing them as 2-Hom-spaces of **linear 2-categories**.

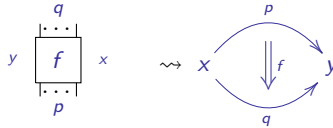
String diagrams

► The 2-cells of a (linear) 2-category can be depicted by a string diagram:

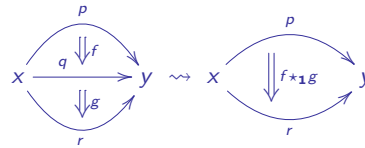
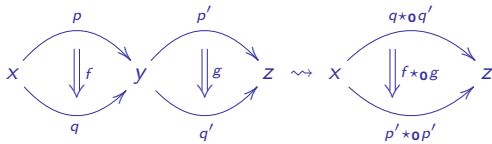


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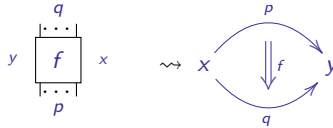


- Compositions:

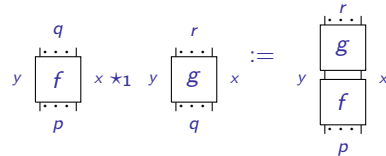
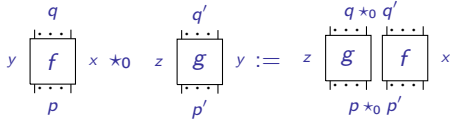
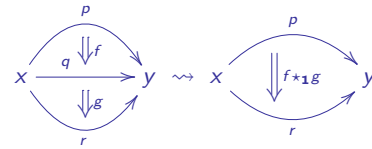
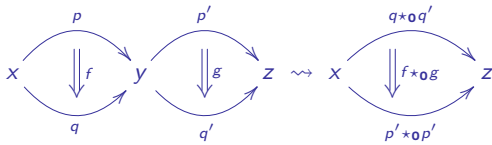


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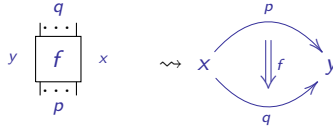


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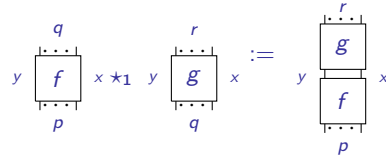
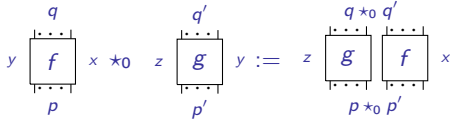
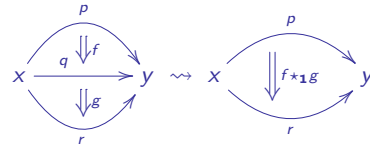
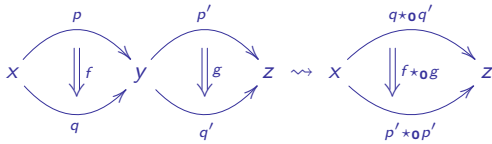


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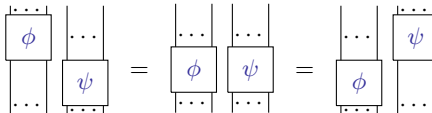
- ▶ The 2-cells of a (linear) 2-category can be depicted by a string diagram:



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- ▶ These compositions satisfy **exchange relations**:



Presentations of linear 2-categories

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satisfying globular relations: $s_0 s_1 = s_0 t_1$, $t_0 s_1 = t_0 t_1$.

- ▶ P_2^* : free 2-category on (P_0, P_1, P_2) .

- ▶ P_2^ℓ : free linear 2-category on (P_0, P_1, P_2) :

$$\forall x, y \in P_1^* : P_2^\ell(x, y) = \mathbb{K}[P_2^*(x, y)].$$

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$$P_3 \begin{array}{c} \xrightarrow{s_2} \\ \xrightarrow{t_2} \end{array} P_2^\ell$$

satisfying $s_1 s_2 = s_1 t_2$, $t_1 s_2 = t_1 t_2$.

- ▶ $P_0 = \{*\}$, $P_1 = \{1 : * \rightarrow *\}$.

- ▶ $P_1^* \simeq \mathbb{N}$ (Number of strands).

- ▶ $P_2 = \{ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} : 2 \rightarrow 2, \quad \begin{array}{c} | \\ | \\ | \end{array} : 1 \rightarrow 1 \}$

- ▶ $P_2^* = \{ \text{diagrams made of } \star_0, \star_1 \text{ compositions of dots and crossings} \}$.

- ▶ $P_2^\ell = \{ \mathbb{K} - \text{linear combinations of diagrams of } P_2^* \}$

Presentations of linear 2-categories

- ▶ Linear 2-categories are presented by generating systems called linear (3,2)-polygraphs, made of a data (P_0, P_1, P_2, P_3) where:

- ▶ (P_1, P_0) is a directed graph, with source and target maps s_0, t_0 .

- ▶ P_1^* : free 1-category generated by (P_0, P_1) .

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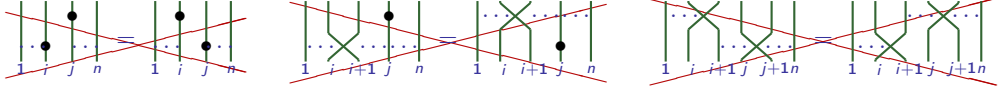
- ▶ P_3 fixes an orientation for the relations of the linear 2-category presented, that is

$$P_2^\ell / \equiv_{P_3} .$$

- ▶ **Example:** For the nil Hecke algebras,

Presentations of linear 2-categories

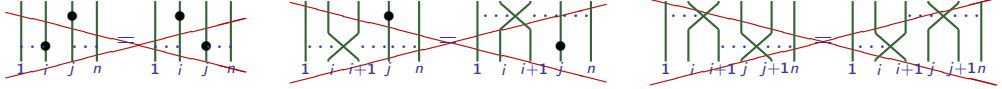
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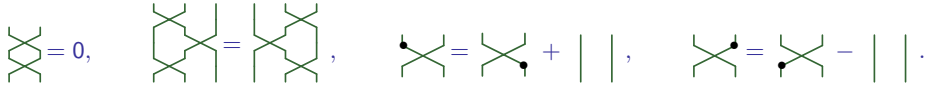
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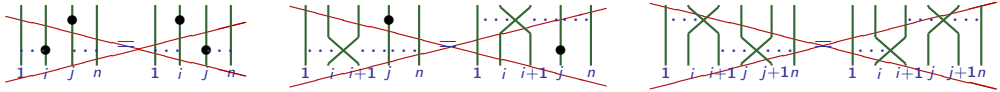
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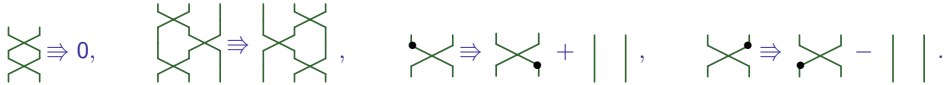
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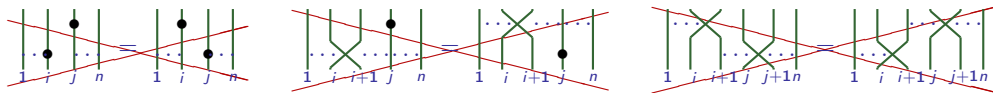


- ▶ This cellular extension defines a linear $(3, 2)$ -polygraph presenting a linear 2-category \mathcal{C} such that

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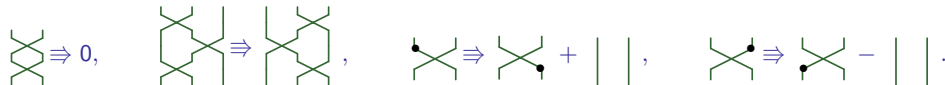
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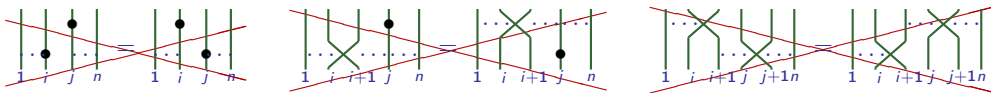
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- ▶ **Assumption:** All the linear $(3, 2)$ -polygraphs we consider are left-monomial.

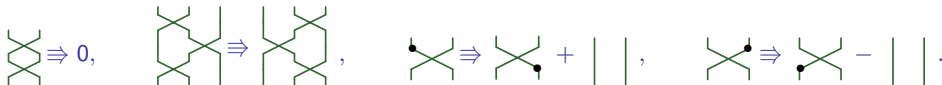
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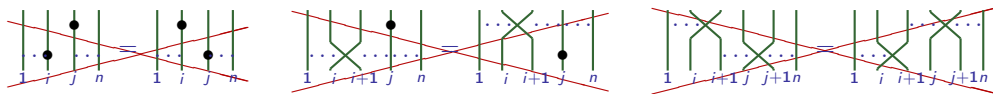
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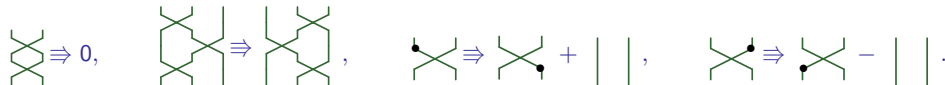
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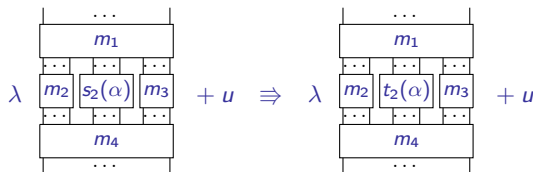
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where $\alpha \in P_3$, and such that $m_1 * \alpha_1 (m_2 * \alpha_0 s_2(\alpha) * \alpha_3) * m_4$ does not appear in the monomial decomposition of u .

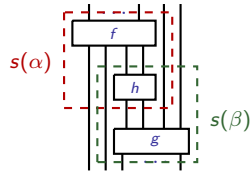
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Bases of linear 2-categories from confluence

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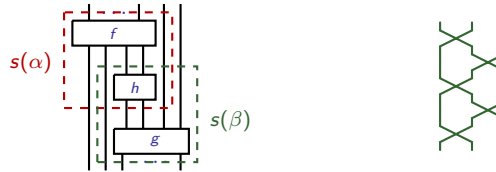
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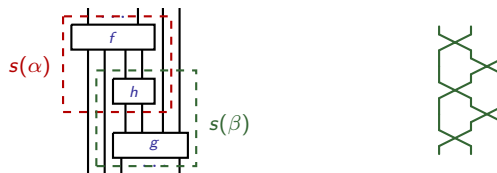


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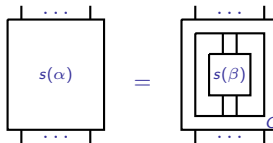
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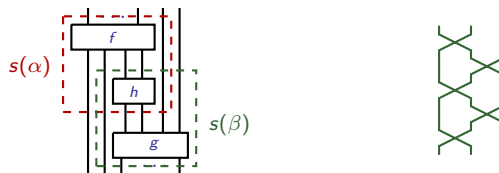


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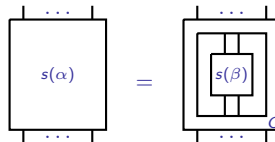
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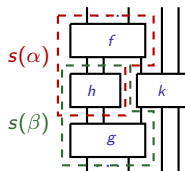
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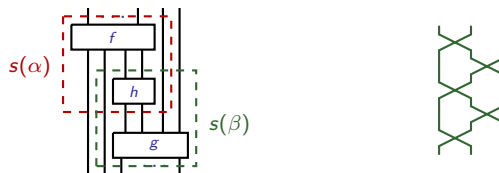


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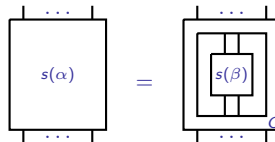
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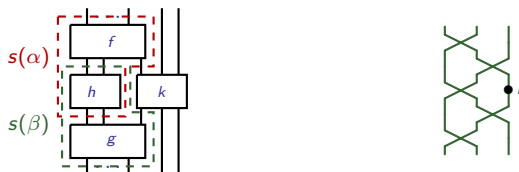
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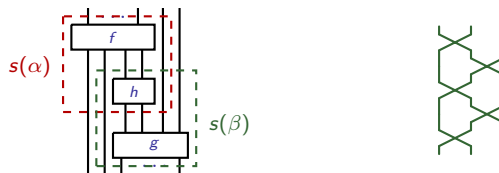


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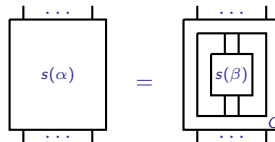
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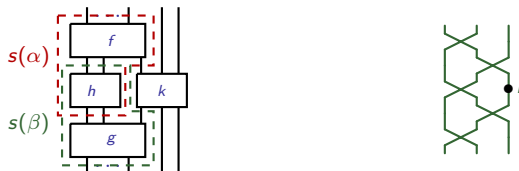
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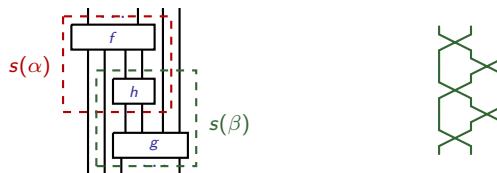
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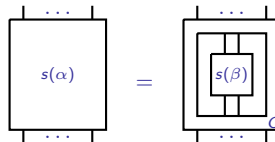
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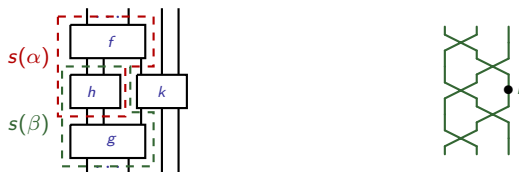
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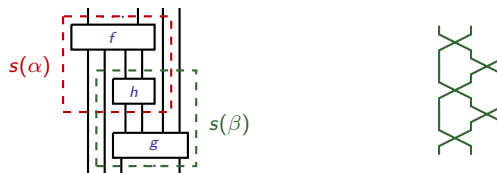
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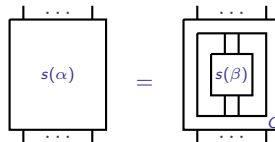
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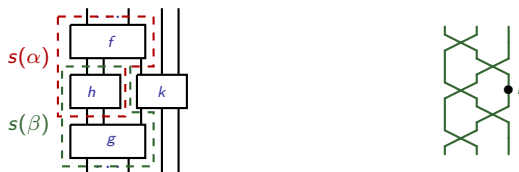
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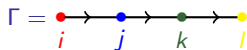
▶ **Theorem [Alleaume '16]:** For any parallel 1-cells p, q of \mathcal{C} , the set of monomials in normal form w.r.t P with 1-source p and 1-target q is a linear basis of $\mathcal{C}_2(p, q)$.

Example: Khovanov-Lauda-Rouquier (KLR) algebras

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- ▶ For such an element \mathcal{V} , we define an algebra $R(\mathcal{V})$.

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- ▶ $R(\mathcal{V})$ is generated by

$$x_{k,i} = \begin{array}{c} | \dots | \bullet | \dots | \\ \color{red}{i_1} \quad \color{blue}{i_k} \quad \color{green}{i_m} \end{array} \quad \text{and} \quad \tau_{k,i} = \begin{array}{c} | \dots | \times | \dots | \\ \color{red}{i_1} \quad \color{blue}{i_\ell} \quad \color{green}{i_{\ell+1}} \quad \color{yellow}{i_m} \end{array}$$

for any $\mathbf{i} = i_1 \dots i_m \in \text{Seq}(\mathcal{V})$, $1 \leq k \leq m$ and $1 \leq \ell < m$.

Convergent presentation of the KLR algebras

► Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: $(\Gamma = \overset{\bullet}{i} \rightarrow \overset{\bullet}{j} \rightarrow \overset{\bullet}{k} \rightarrow \overset{\bullet}{l})$

i) Same color:

$$\text{Red crossing} = 0 \quad \text{Red crossing with dot on top} = \text{Red crossing with dot on bottom} + \text{Two vertical lines}, \quad \text{Red crossing with dot on bottom} = \text{Red crossing with dot on top} - \text{Two vertical lines}$$

ii) Distant colors:

$$\text{Red and green crossing} = \text{Two vertical lines}$$

iii) Close colors:

$$\text{Red and blue crossing} = \text{Red dot on top} + \text{Blue dot on top}$$

iv) Different colors:

$$\text{Blue and green crossing} = \text{Green and blue crossing}, \quad \text{Green and blue crossing with dot on top} = \text{Green and blue crossing with dot on bottom}$$

vi) Braid relations:

$$\text{Red and blue braid} = \text{Red and blue braid} + \text{Two vertical lines} \quad \text{and otherwise} \quad \text{Green and yellow braid} = \text{Green and yellow braid}$$

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iii) Close colors:

$$\begin{array}{c} \color{blue} \text{X} \\ \color{red} \end{array} \Rightarrow \begin{array}{c} \bullet \\ \color{red} \parallel \end{array} + \begin{array}{c} \color{red} \parallel \\ \bullet \end{array}$$

iv) Different colors:

$$\begin{array}{c} \color{blue} \text{X} \\ \color{green} \end{array} \Rightarrow \begin{array}{c} \color{green} \text{X} \\ \color{blue} \end{array}, \quad \begin{array}{c} \color{green} \text{X} \\ \bullet \end{array} \Rightarrow \bullet \color{green} \text{X}$$

vi) Braid relations:

$$\begin{array}{c} \color{blue} \text{X} \\ \color{red} \end{array} \Rightarrow \begin{array}{c} \color{red} \text{X} \\ \color{blue} \end{array} + \color{red} \parallel \color{blue} \parallel \quad \text{and otherwise} \quad \begin{array}{c} \color{green} \text{X} \\ \color{yellow} \end{array} \Rightarrow \begin{array}{c} \color{yellow} \text{X} \\ \color{green} \end{array}$$

Convergent presentation of the KLR algebras

► Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: $(\Gamma = \overset{\bullet}{i} \rightarrow \overset{\bullet}{j} \rightarrow \overset{\bullet}{k} \rightarrow \overset{\bullet}{l})$

i) Same color:

$$\begin{array}{c} \text{Red crossing} \\ \Rightarrow 0 \end{array} \quad \begin{array}{c} \text{Red crossing with dot on top-left} \\ \Rightarrow \text{Red crossing with dot on top-right} + \text{Two vertical red lines} \end{array}, \quad \begin{array}{c} \text{Red crossing with dot on top-right} \\ \Rightarrow \text{Red crossing with dot on top-left} - \text{Two vertical red lines} \end{array}$$

ii) Distant colors:

$$\begin{array}{c} \text{Red and green crossing} \\ \Rightarrow \text{Two vertical lines (red and green)} \end{array}$$

iii) Close colors:

$$\begin{array}{c} \text{Red and blue crossing} \\ \Rightarrow \text{Red line with dot} + \text{Blue line with dot} \end{array}$$

iv) Different colors:

$$\begin{array}{c} \text{Blue and green crossing} \\ \Rightarrow \text{Blue and green crossing (swapped)} \end{array} \quad \begin{array}{c} \text{Blue and green crossing with dot on top-left} \\ \Rightarrow \text{Blue and green crossing with dot on top-right} \end{array}$$

vi) Braid relations:

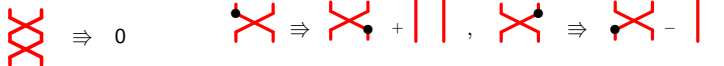
$$\begin{array}{c} \text{Blue and red crossing} \\ \Rightarrow \text{Blue and red crossing (swapped)} + \text{Two vertical lines (blue and red)} \end{array} \quad \text{and otherwise} \quad \begin{array}{c} \text{Green and yellow crossing} \\ \Rightarrow \text{Green and yellow crossing (swapped)} \end{array}$$

► **Theorem [D. '19]:** This linear $(3, 2)$ -polygraph is convergent.

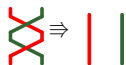
Convergent presentation of the KLR algebras

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i) Same color:



ii) Distant colors:



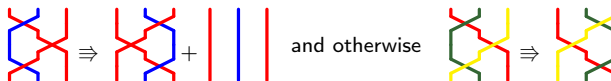
iii) Close colors:



iv) Different colors:



vi) Braid relations:



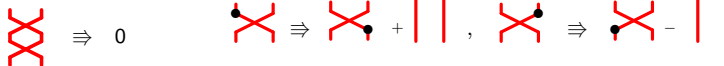
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► Idea for termination: number of crossings is decreasing, permutations are left adjusted and dots move to the bottom.

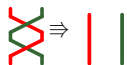
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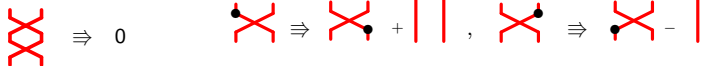
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- Confluence: exhaustive study of all critical branchings.

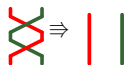
Convergent presentation of the KLR algebras

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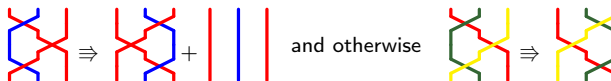
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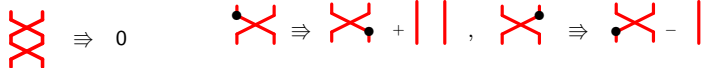
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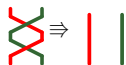
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i) Same color:

$$\begin{array}{c} \text{Red crossing} \\ \Downarrow \\ 0 \end{array} \quad \begin{array}{c} \text{Red crossing with dot on top-left} \\ \Downarrow \\ \text{Red crossing with dot on top-right} + \text{Two vertical lines} \end{array}, \quad \begin{array}{c} \text{Red crossing with dot on top-right} \\ \Downarrow \\ \text{Red crossing with dot on top-left} - \text{Two vertical lines} \end{array}$$

ii) Distant colors:

$$\begin{array}{c} \text{Crossing of red and green lines} \\ \Downarrow \\ \text{Two vertical lines (one red, one green)} \end{array}$$

iii) Close colors:

$$\begin{array}{c} \text{Crossing of blue and red lines} \\ \Downarrow \\ \text{Red line with dot} + \text{Blue line with dot} \end{array}$$

iv) Different colors:

$$\begin{array}{c} \text{Crossing of blue and green lines} \\ \Downarrow \\ \text{Blue line with dot} + \text{Green line with dot} \end{array}$$

vi) Braid relations:

$$\begin{array}{c} \text{Braid relation 1} \\ \Downarrow \\ \text{Braid relation 2} + \text{Two vertical lines} \end{array} \quad \text{and otherwise} \quad \begin{array}{c} \text{Braid relation 3} \\ \Downarrow \\ \text{Braid relation 4} \end{array}$$

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- Idea for termination: number of crossings is decreasing, permutations are left adjusted and dots move to the bottom.
- Confluence: exhaustive study of all critical branchings.

$$\begin{array}{c} \text{Complex crossing} \\ \Downarrow \\ \text{Sum of two simpler crossings} \end{array}$$

Convergent presentation of the KLR algebras

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i) Same color:

$$\begin{array}{c} \text{Red crossing} \\ \Downarrow \\ 0 \end{array} \quad \begin{array}{c} \text{Red crossing with dot on top-left} \\ \Downarrow \\ \text{Red crossing with dot on bottom-right} + \text{Two vertical red lines} \end{array}, \quad \begin{array}{c} \text{Red crossing with dot on top-right} \\ \Downarrow \\ \text{Red crossing with dot on bottom-left} - \text{Two vertical red lines} \end{array}$$

ii) Distant colors:

$$\begin{array}{c} \text{Crossing of red and green lines} \\ \Downarrow \\ \text{Two vertical lines, one red, one green} \end{array}$$

iii) Close colors:

$$\begin{array}{c} \text{Crossing of blue and red lines} \\ \Downarrow \\ \text{Red line with dot on top} + \text{Blue line with dot on top} \end{array}$$

iv) Different colors:

$$\begin{array}{c} \text{Crossing of blue and green lines} \\ \Downarrow \\ \text{Blue line with dot on top} \end{array} \quad \begin{array}{c} \text{Crossing of blue and green lines} \\ \Downarrow \\ \text{Green line with dot on top} \end{array}$$

vi) Braid relations:

$$\begin{array}{c} \text{Braid relation between blue and red lines} \\ \Downarrow \\ \text{Braid relation between red and blue lines} + \text{Two vertical lines (one blue, one red)} \end{array} \quad \text{and otherwise} \quad \begin{array}{c} \text{Braid relation between green and yellow lines} \\ \Downarrow \\ \text{Braid relation between yellow and green lines} \end{array}$$

► **Theorem [D. '19]:** This linear $(3, 2)$ -polygraph is convergent.

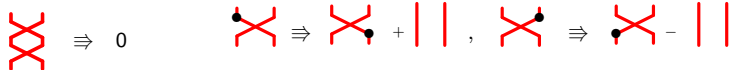
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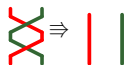
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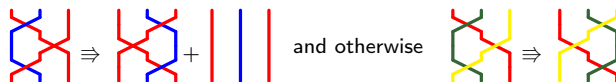
iii) Close colors:



iv) Different colors:

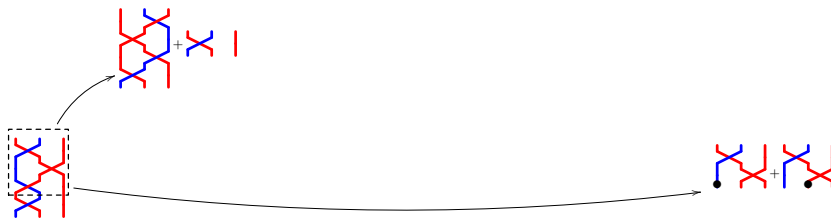


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i) Same color:

$$\begin{array}{c} \text{XX} \\ \Rightarrow 0 \end{array} \quad \begin{array}{c} \bullet \\ \text{XX} \\ \Rightarrow \text{XX} \bullet + \text{||} \end{array}, \quad \begin{array}{c} \text{XX} \\ \bullet \\ \Rightarrow \bullet \text{XX} - \text{||} \end{array}$$

ii) Distant colors:

$$\begin{array}{c} \text{XX} \\ \text{YY} \\ \Rightarrow \text{||} \text{||} \end{array}$$

iii) Close colors:

$$\begin{array}{c} \text{XX} \\ \text{YY} \\ \Rightarrow \bullet \text{||} + \text{||} \bullet \end{array}$$

iv) Different colors:

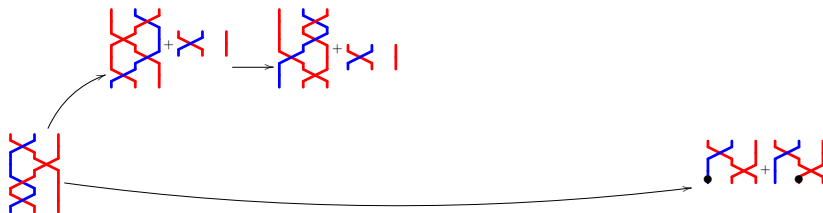
$$\begin{array}{c} \bullet \\ \text{XY} \\ \Rightarrow \text{XY} \bullet \end{array} \quad \begin{array}{c} \text{XY} \\ \bullet \\ \Rightarrow \bullet \text{XY} \end{array}$$

vi) Braid relations:

$$\begin{array}{c} \text{XY} \\ \text{XY} \\ \Rightarrow \text{XYXY} + \text{||} \text{||} \end{array} \quad \text{and otherwise} \quad \begin{array}{c} \text{XY} \\ \text{XY} \\ \Rightarrow \text{XYXY} \end{array}$$

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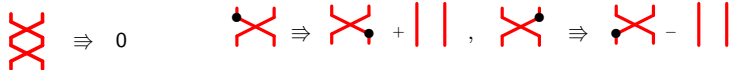
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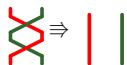
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i) Same color:



ii) Distant colors:



iii) Close colors:



iv) Different colors:

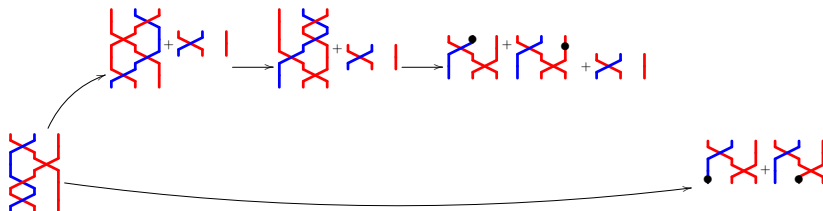


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i) Same color:

$$\begin{array}{c} \text{X} \\ \text{X} \end{array} \Rightarrow 0 \quad \begin{array}{c} \bullet \\ \text{X} \end{array} \Rightarrow \begin{array}{c} \text{X} \\ \bullet \end{array} + \begin{array}{|l} | \\ | \\ | \end{array}, \quad \begin{array}{c} \text{X} \\ \bullet \end{array} \Rightarrow \begin{array}{c} \bullet \\ \text{X} \end{array} - \begin{array}{|l} | \\ | \\ | \end{array}$$

ii) Distant colors:

$$\begin{array}{c} \text{X} \\ \text{Y} \end{array} \Rightarrow \begin{array}{|l} | \\ | \end{array}$$

iii) Close colors:

$$\begin{array}{c} \text{X} \\ \text{Y} \end{array} \Rightarrow \begin{array}{c} \bullet \\ | \end{array} + \begin{array}{c} | \\ \bullet \end{array}$$

iv) Different colors:

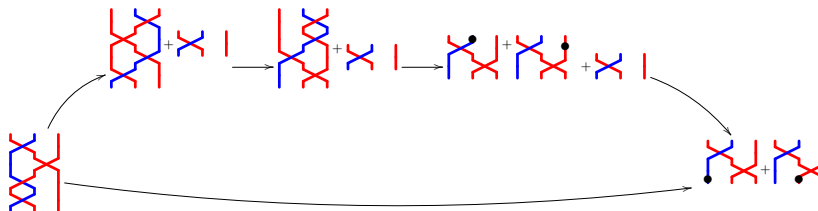
$$\begin{array}{c} \text{Y} \\ \text{X} \end{array} \Rightarrow \begin{array}{c} \text{X} \\ \text{Y} \end{array}, \quad \begin{array}{c} \text{X} \\ \text{Y} \end{array} \Rightarrow \begin{array}{c} \text{Y} \\ \text{X} \end{array}$$

vi) Braid relations:

$$\begin{array}{c} \text{X} \\ \text{Y} \end{array} \begin{array}{c} \text{X} \\ \text{Y} \end{array} \Rightarrow \begin{array}{c} \text{X} \\ \text{Y} \end{array} \begin{array}{c} \text{Y} \\ \text{X} \end{array} + \begin{array}{|l} | \\ | \\ | \end{array} \quad \text{and otherwise} \quad \begin{array}{c} \text{Y} \\ \text{X} \end{array} \begin{array}{c} \text{Y} \\ \text{X} \end{array} \Rightarrow \begin{array}{c} \text{Y} \\ \text{X} \end{array} \begin{array}{c} \text{X} \\ \text{Y} \end{array}$$

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III. Confluence modulo in the KLR 2-category

- ▶ Proving confluence for presentations admitting a great number of relations may be complicated.

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Rewriting modulo

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 - ▶ **Example:** Adjunction and isotopy relations in pivotal linear 2-categories:

$$\begin{array}{c} \text{cap} \\ \text{cup} \end{array} = | = \begin{array}{c} \text{cup} \\ \text{cap} \end{array}, \quad \begin{array}{c} \text{cup} \\ \bullet \end{array} = | = \begin{array}{c} \text{cap} \\ \bullet \end{array}, \quad \begin{array}{c} \text{cap} \\ \bullet \end{array} = \begin{array}{c} \text{cup} \\ \bullet \end{array}, \quad \begin{array}{c} \text{cup} \\ \bullet \end{array} = \begin{array}{c} \text{cup} \\ \bullet \end{array}.$$

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- ▶ **Example:** Adjunction and isotopy relations in pivotal linear 2-categories:

The diagram shows four pairs of equations representing adjunction and isotopy relations in pivotal linear 2-categories. Each pair is separated by a comma. The first pair shows a cup (two arcs meeting at a top point) equal to a vertical line equal to a cap (two arcs meeting at a bottom point). The second pair shows a cup with a dot on its top arc equal to a vertical line with a dot equal to a cap with a dot on its bottom arc. The third pair shows a cap with a dot on its top arc equal to a cap with a dot on its bottom arc. The fourth pair shows a cup with a dot on its bottom arc equal to a cup with a dot on its top arc.

- ▶ **Rewriting modulo** these relations: R set of oriented relations and E set of non-oriented axioms.

Rewriting modulo

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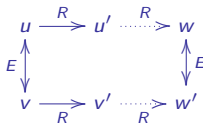
- ▶ **Example:** Adjunction and isotopy relations in pivotal linear 2-categories:

$$\begin{array}{c} \text{cap} \\ \text{cup} \end{array} = \text{dot} = \begin{array}{c} \text{cup} \\ \text{cap} \end{array}, \quad \begin{array}{c} \text{cap} \\ \text{cup} \end{array} = \text{dot} = \begin{array}{c} \text{cup} \\ \text{cap} \end{array}, \quad \begin{array}{c} \text{cap} \\ \text{cup} \end{array} = \begin{array}{c} \text{cap} \\ \text{cup} \end{array}, \quad \begin{array}{c} \text{cup} \\ \text{cap} \end{array} = \begin{array}{c} \text{cup} \\ \text{cap} \end{array}.$$

- ▶ **Rewriting modulo** these relations: R set of oriented relations and E set of non-oriented axioms.

- ▶ Three main paradigms of rewriting modulo:

- ▶ Rewriting with relations of R , and confluence modulo E , **Huet '80**.



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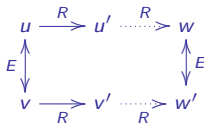
- ▶ **Example:** Adjunction and isotopy relations in pivotal linear 2-categories:

$$\begin{array}{c} \cup \\ \cup \end{array} = | = \begin{array}{c} \cup \\ \cup \end{array}, \quad \begin{array}{c} \cup \\ \cup \end{array} = | = \begin{array}{c} \cup \\ \cup \end{array}, \quad \begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array}, \quad \begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array}.$$

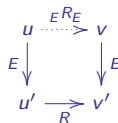
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- ▶ Rewriting with R on E -equivalence classes:



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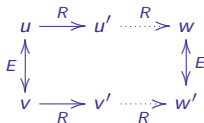
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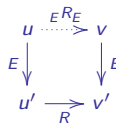
- ▶ **Rewriting modulo** these relations: R set of oriented relations and E set of non-oriented axioms.

- ▶ Three main paradigms of rewriting modulo:

- ▶ Rewriting with relations of R , and confluence modulo E , **Huet '80**.



- ▶ Rewriting with R on E -equivalence classes:



- ▶ **Rewriting system modulo:** (R, E, S) such that $R \subseteq S \subseteq ER_E$, **Jouannaud-Kirchner '84**.

Linear $(3, 2)$ -polygraphs modulo

- ▶ We introduce a polygraphic setting for rewriting modulo in diagrammatic linear 2-categories.

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 - ▶ a linear $(3, 2)$ -polygraph R ,

Linear $(3, 2)$ -polygraphs modulo

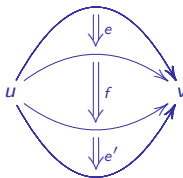
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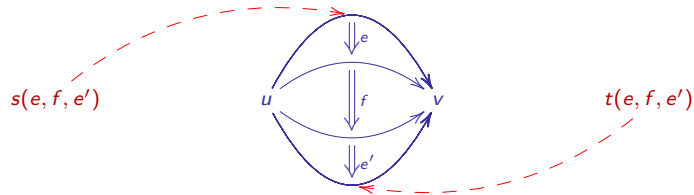
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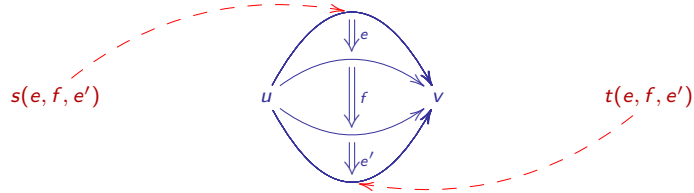
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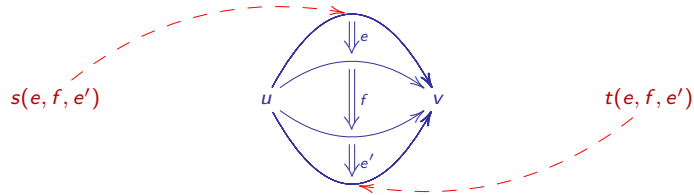
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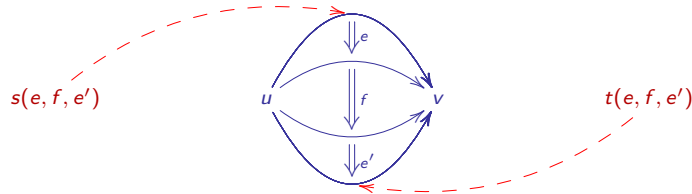


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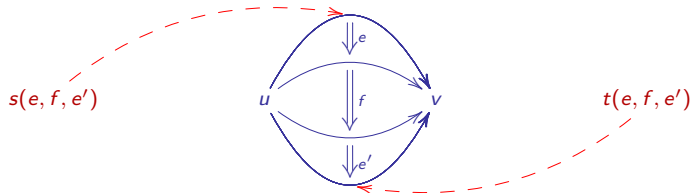
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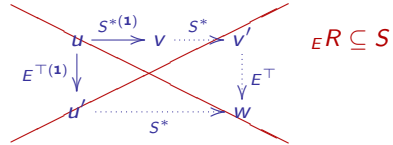
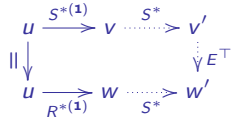
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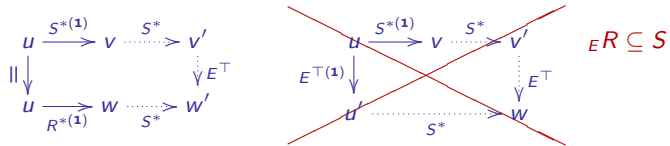
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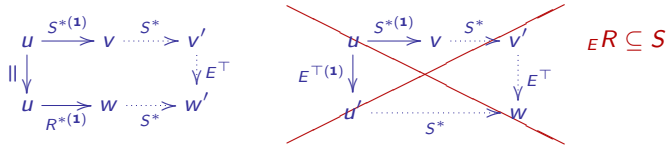


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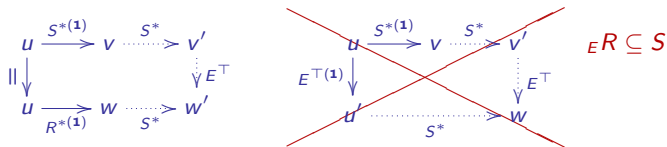


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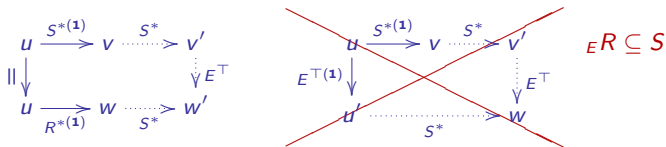


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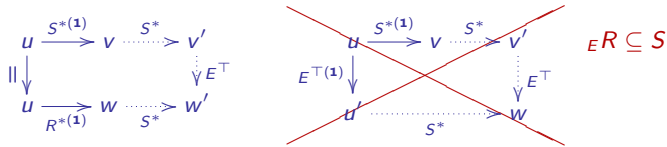
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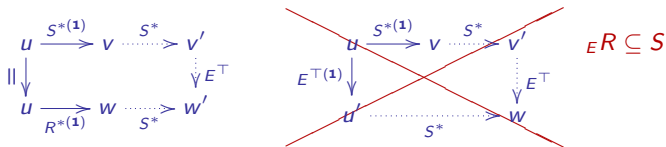
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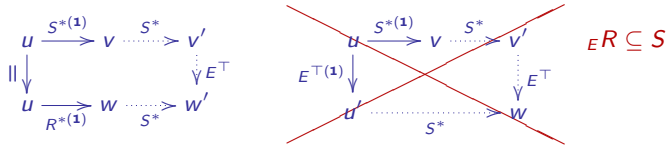
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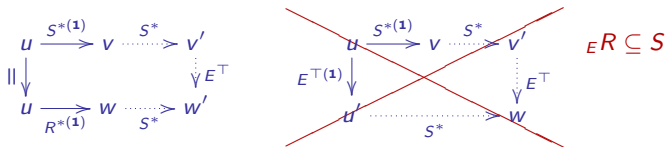
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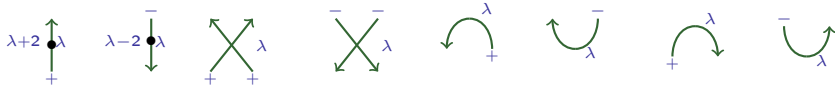
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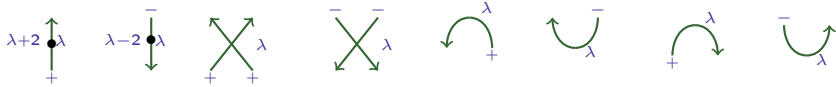
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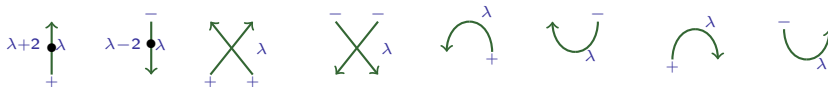
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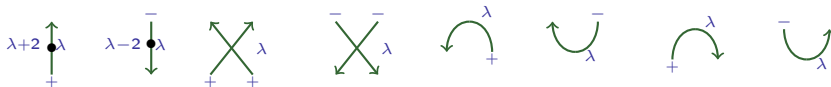


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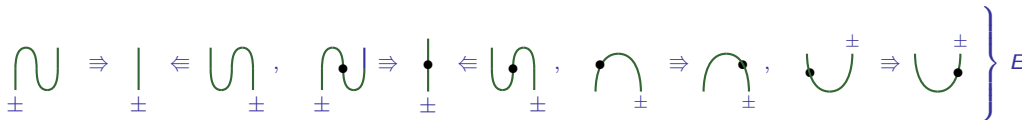
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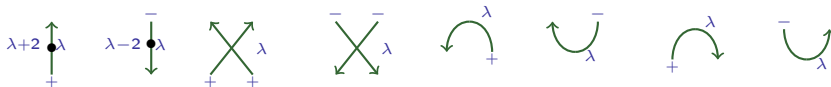
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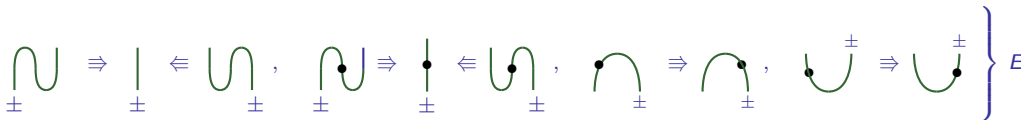
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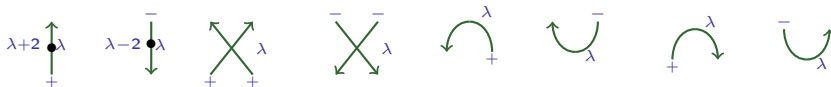
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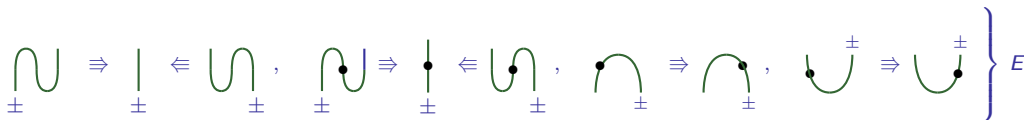
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$$n \begin{array}{c} \circlearrowleft \\ \lambda \end{array} \equiv \begin{cases} 1_{1_\lambda} & \text{if } n = \lambda - 1 \\ 0 & \text{if } n < \lambda - 1 \end{cases} ; \quad \lambda \begin{array}{c} \circlearrowright \\ n \end{array} \equiv \begin{cases} 1_{1_\lambda} & \text{if } n = -\lambda - 1 \\ 0 & \text{if } n < -\lambda - 1 \end{cases}$$

$$\lambda - 1 + \alpha \begin{array}{c} \circlearrowleft \\ \lambda \end{array} \equiv - \sum_{f=1}^{\alpha} \lambda - 1 + \alpha - f \begin{array}{c} \circlearrowleft \\ \lambda \end{array} \begin{array}{c} \circlearrowright \\ \lambda - \lambda - 1 + f \end{array} \text{ for all } \lambda \in \mathbb{Z} \text{ and } \alpha > 0 \text{ such that } \lambda - 1 + \alpha \geq 0$$

► Bubble slide relations of the form

$$\lambda + 1 + \alpha \begin{array}{c} \circlearrowleft \\ \lambda \end{array} \uparrow \equiv \sum_{f=0}^{\alpha} (\alpha + 1 - f) \uparrow \begin{array}{c} \circlearrowright \\ \lambda \end{array} \begin{array}{c} \lambda - 1 + f \end{array}$$

for any orientations of the bubbles and of the strand.

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Non degeneracy of Khovanov-Lauda's diagrammatic calculus

► **Corollary:** A fixed set of quasi-normal forms containing diagrams with source p and target q in normal form with respect to E and having:

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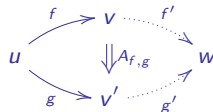
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IV. Conclusion and perspectives

- ▶ We developed effective tools based on rewriting modulo to compute in (linear) diagrammatic presentations.

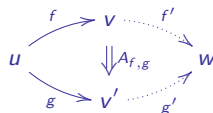
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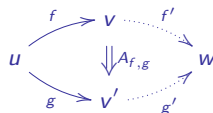
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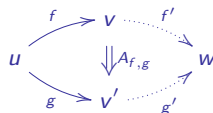


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- ▶ **Objective:** extend these constructions in higher dimensions.

- ▶ Work in progress:

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1.) Extension of these methods to the case of monoidal supercategories and 2-supercategories, work in progress with **M. Ebert** and **A. Lauda**.

- Introduction of $(3, 2)$ -superpolygraphs and of super rewriting theory using implicit rewriting modulo **super-exchange laws**:

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- **Long-term project:** Implement computational tools to analyse confluence of diagrammatic presentations.

Thank you for your attention.