Réécriture modulo dans les catégories diagrammatiques.

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Institut Camille Jordan, Université Lyon 1

Soutenance de thèse de Doctorat

Sous la direction de Philippe Malbos, Stéphane Gaussent et Alistair Savage

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Outline

I. Introduction

Linear bases using rewriting

- Algebraic rewriting: model of computations for algebraic structures presented by generators and oriented relations.

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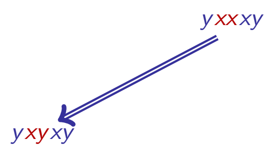

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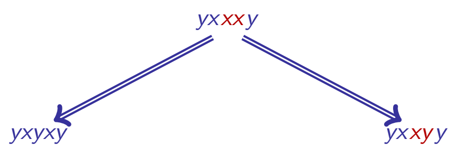

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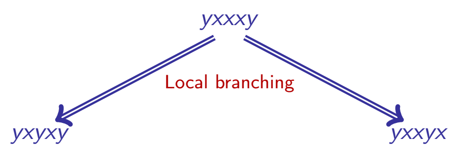

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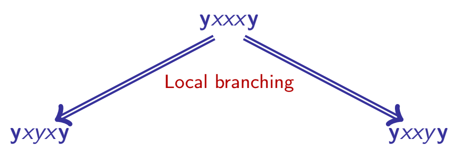

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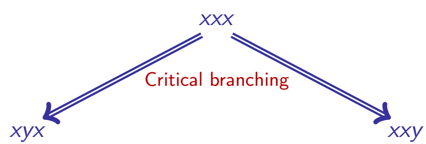

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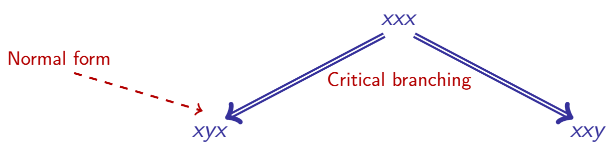

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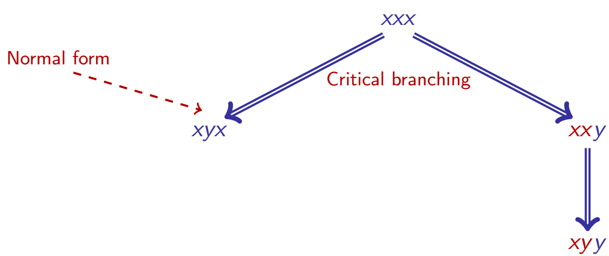

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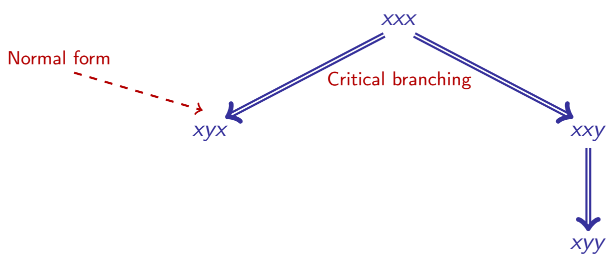

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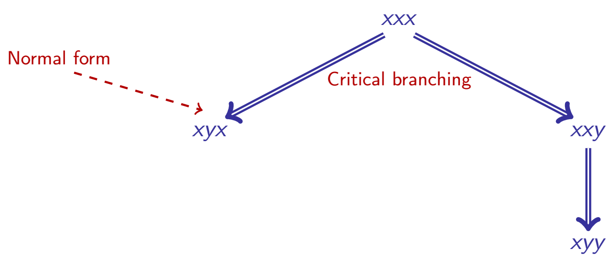

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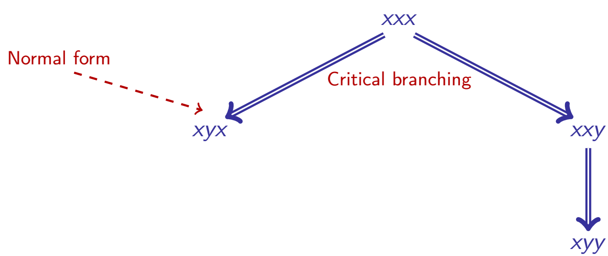

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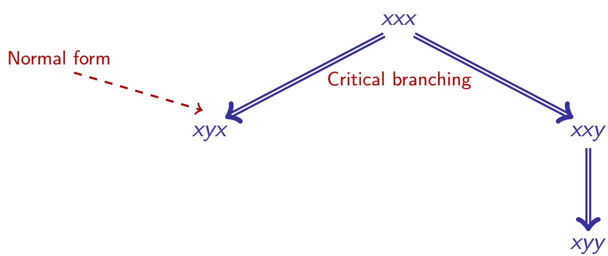

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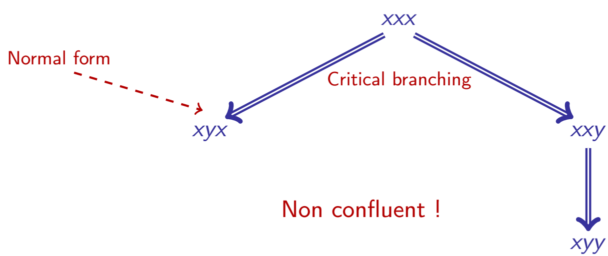

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  i) P is confluent.

  ii) The vector space $P_1^{\ell} := \mathbb{K}[P_1^*]$ admits the direct decomposition

  $$P_1^{\ell} = P_1^{\mathrm{nf}} \oplus I(P)$$

  where P_1^{nf} is the set of monomials in normal form with respect to P_1, and $I(P)$ is the two sided ideal generated by $\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in P_2\}$.

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  - Develop linear rewriting modulo theory to compute linear bases from confluence modulo.
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Rewriting in categorification theory

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- Several new questions, e.g. extension to rewriting in 2-supercategories (with **M. Ebert** and **A. Lauda**) and explicit proofs of categorification (with **G. Naisse**).

II. Convergent presentation of the KLR algebras

Diagrammatic algebras and categories

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  - relations:

$$x_i x_j = x_j x_i$$
$$\tau_i x_j = x_j \tau_i \quad \text{if } |i-j| > 1$$
$$\tau_i \tau_j = \tau_j \tau_i \quad \text{if } |i-j| > 1$$
$$\tau_i^2 = 0$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$$
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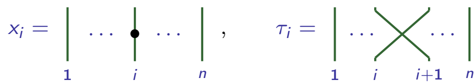

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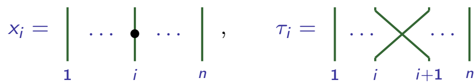

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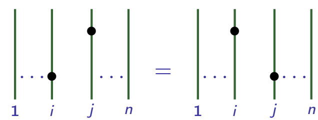

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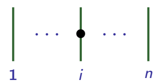
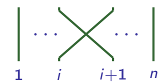
$x_i =$ 

, $\tau_i =$ 

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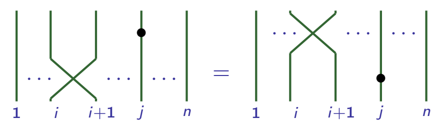

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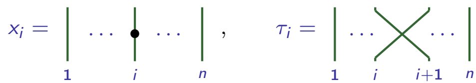

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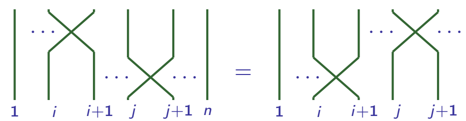

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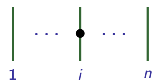
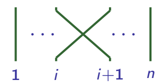
$x_i =$ 

, $\tau_i =$ 

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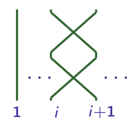

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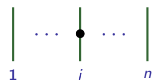
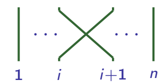
$x_i =$ 

, $\tau_i =$ 

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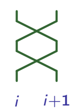

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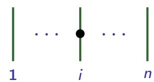
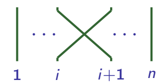
$x_i =$ 

, $\tau_i =$ 

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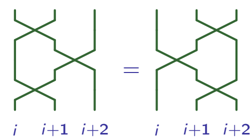

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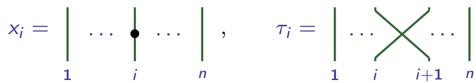

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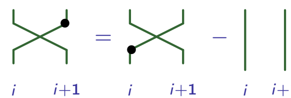

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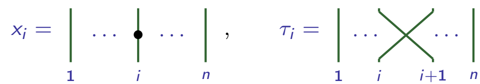

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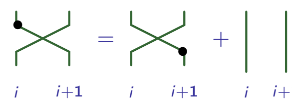

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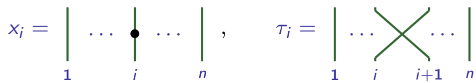

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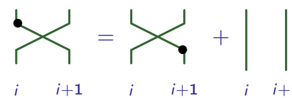
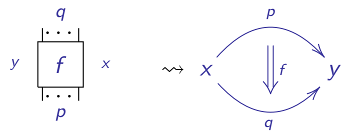


- We study these algebras by realizing them as 2-Hom-spaces of linear 2-categories.

String diagrams

- The 2-cells of a (linear) 2-category can be depicted by a string diagram:



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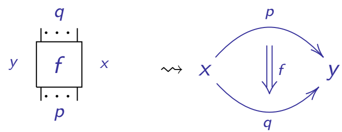

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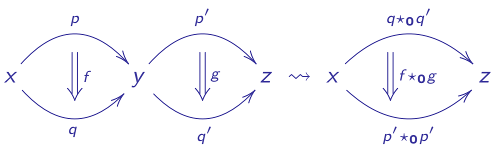

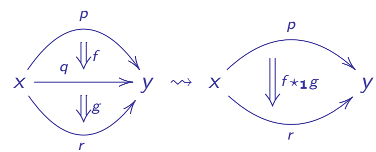

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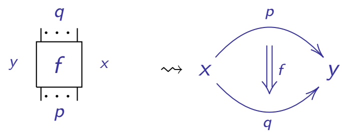

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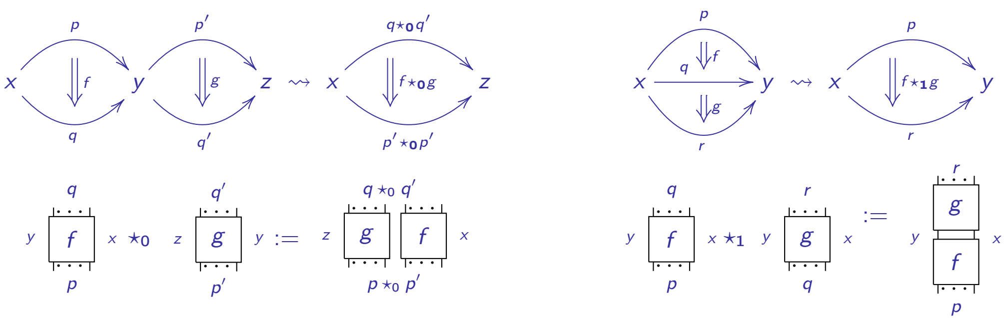

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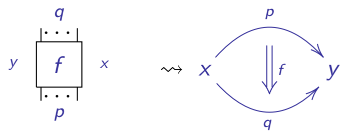

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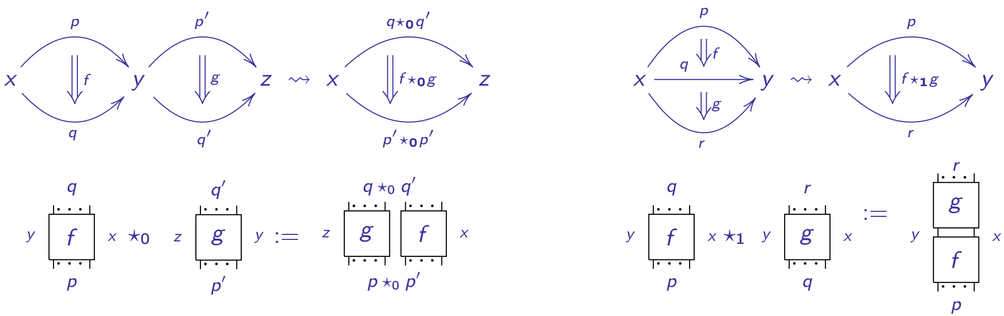

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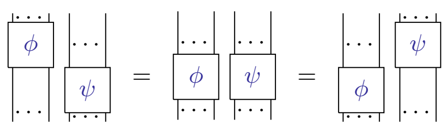

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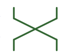
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$

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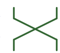
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$

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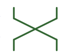
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$
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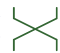
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$
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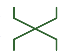
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$
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Presentations of linear 2-categories

- Linear 2-categories are presented by generating systems called linear $(3,2)$-polygraphs, made of a data (P_0, P_1, P_2, P_3) where:

- (P_1, P_0) is a directed graph, with source and target maps s_0, t_0.
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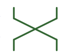
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$
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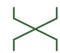
- $P_2 = \{$  $: 2 \to 2,$ $: 1 \to 1\}$
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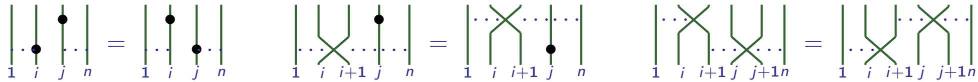

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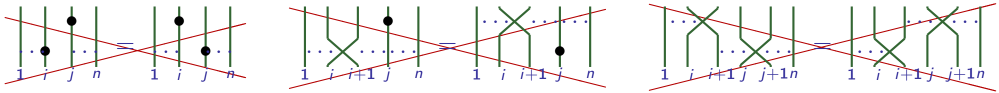

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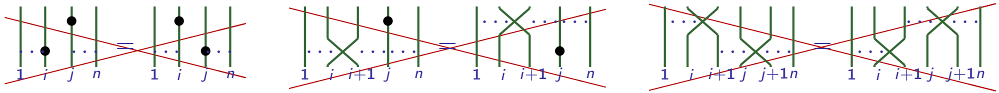

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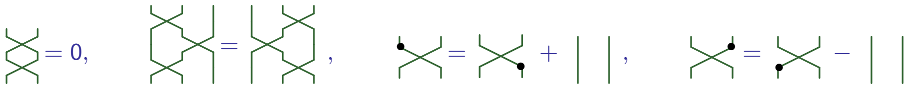

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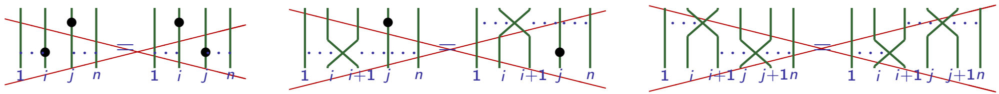

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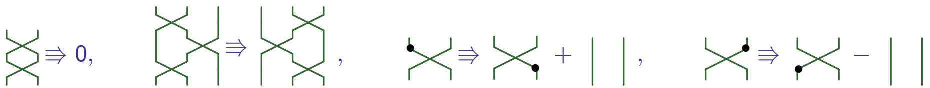

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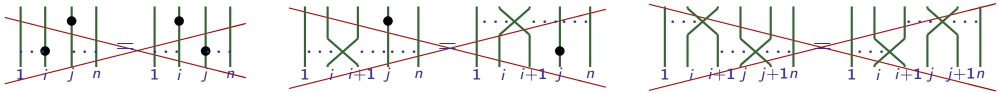

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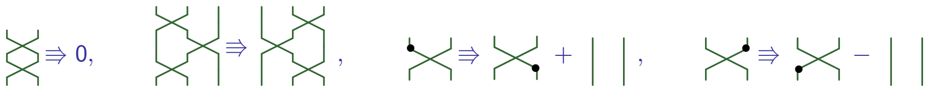

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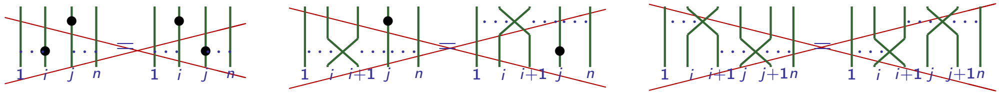

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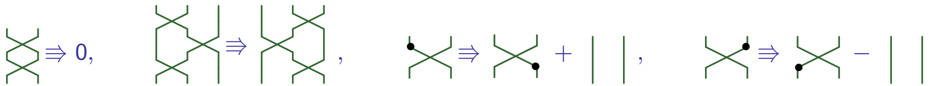

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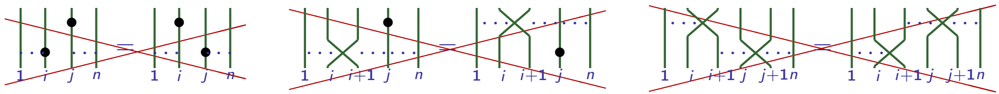

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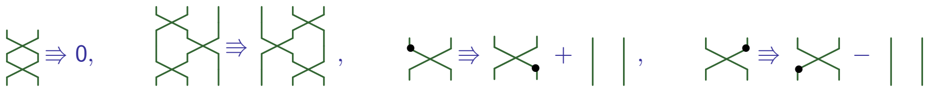

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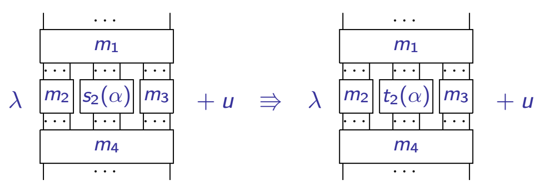

where $\alpha \in P_3$, and such that $m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4$ does not appear in the monomial decomposition of u.

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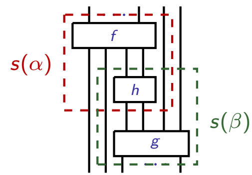

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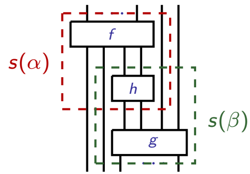

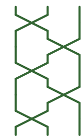

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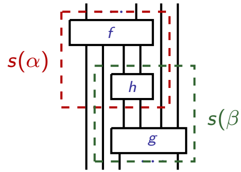

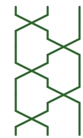

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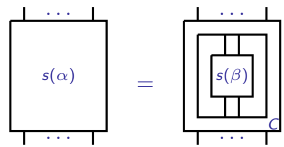

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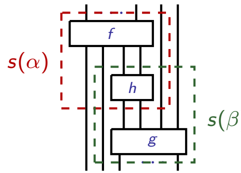

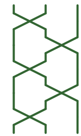

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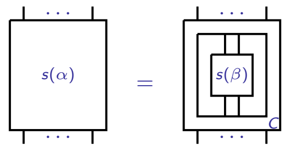

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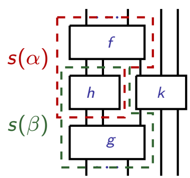

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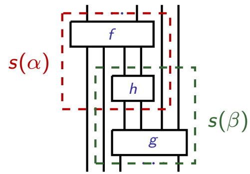

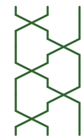

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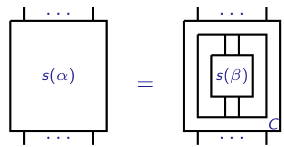

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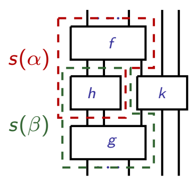

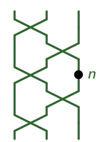

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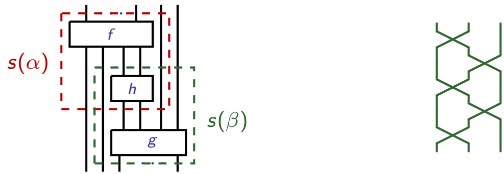

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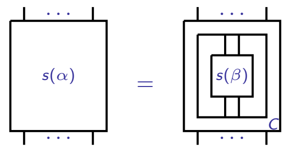

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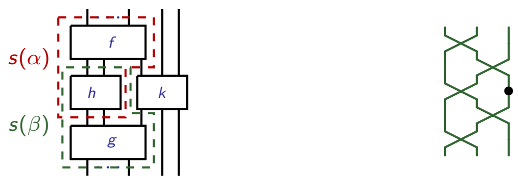

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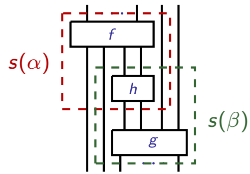

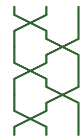

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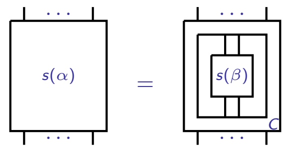

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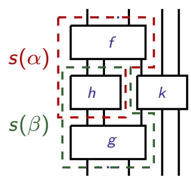

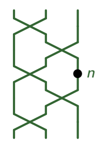

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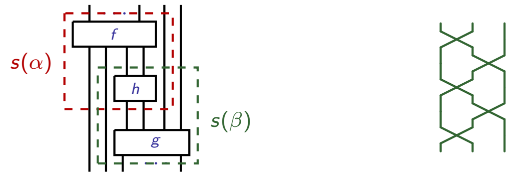

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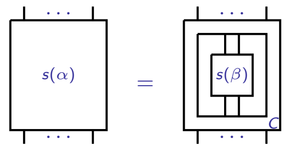

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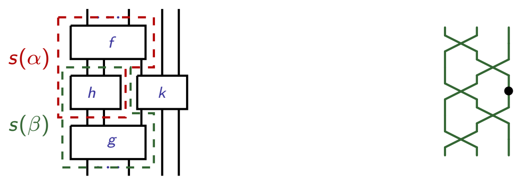

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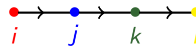
$\Gamma =$ 

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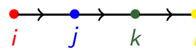
$\Gamma =$ 

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$$\Gamma = \underset{i}{\bullet} \longrightarrow \underset{j}{\bullet} \longrightarrow \underset{k}{\bullet} \longrightarrow \underset{l}{\bullet} \qquad (\Gamma \text{ simply laced})$$

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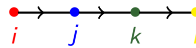
$\Gamma =$ 

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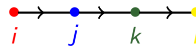
$\Gamma =$ 

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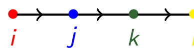
$\Gamma =$  $(\Gamma$ simply laced$)$

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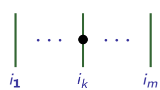
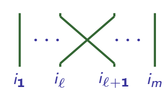
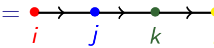
- $R(\mathcal{V})$ is generated by

$x_{k,\mathbf{i}} =$  and $\tau_{k,\mathbf{i}} =$ 

for any $\mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$, $1 \le k \le m$ and $1 \le \ell < m$.

Convergent presentation of the KLR algebras

- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

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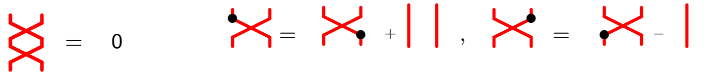

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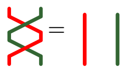

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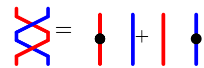

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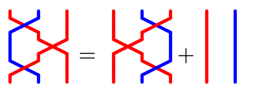

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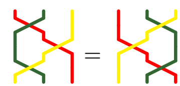

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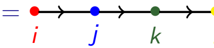
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

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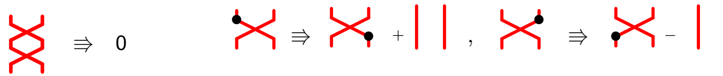

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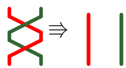

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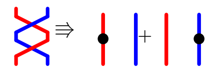

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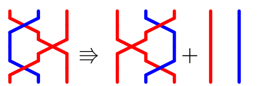

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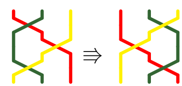

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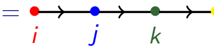
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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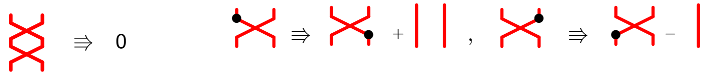

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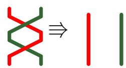

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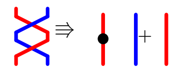

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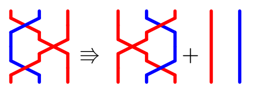

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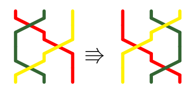

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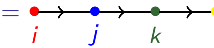
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

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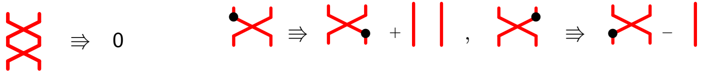

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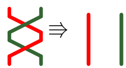

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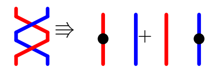

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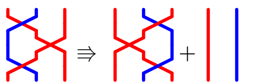
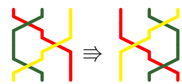
 and otherwise 

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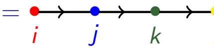
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ 

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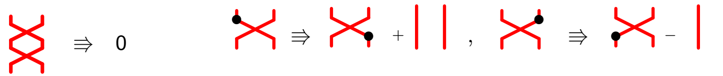

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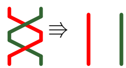

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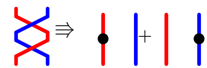

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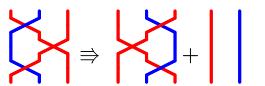
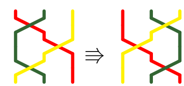
 and otherwise 

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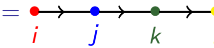
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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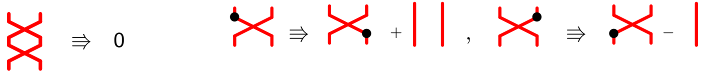

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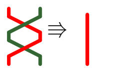

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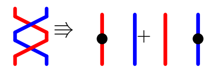

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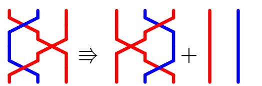
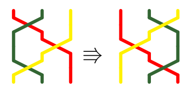
 and otherwise 

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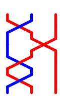

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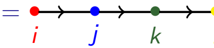
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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  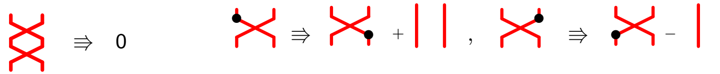

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  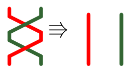

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  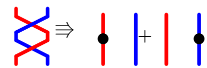

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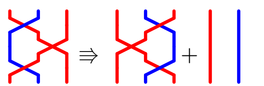
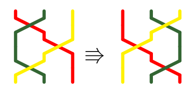
   and otherwise 

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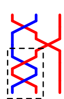

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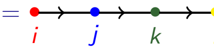
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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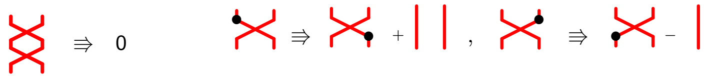

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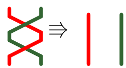

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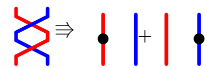

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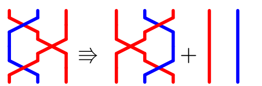
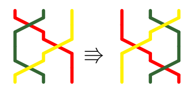
 and otherwise 

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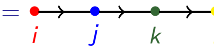
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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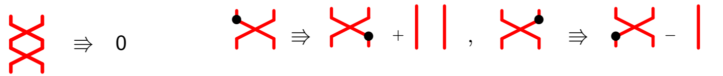

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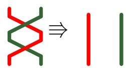

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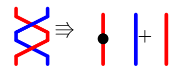

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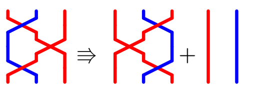
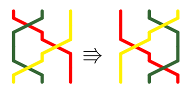
 and otherwise 

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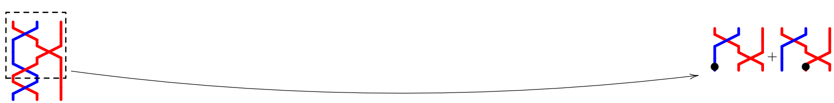

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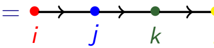
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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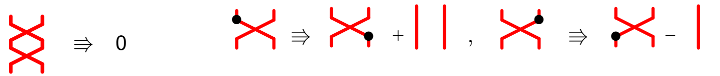

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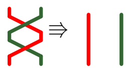

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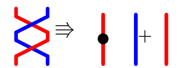

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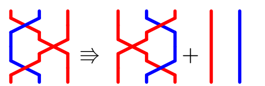

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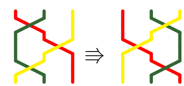

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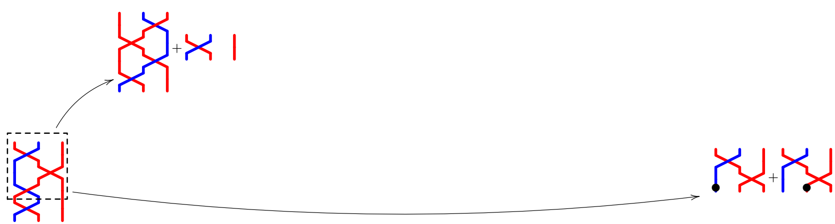

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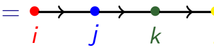
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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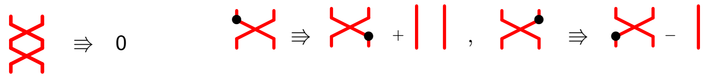

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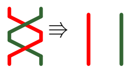

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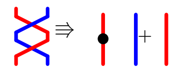

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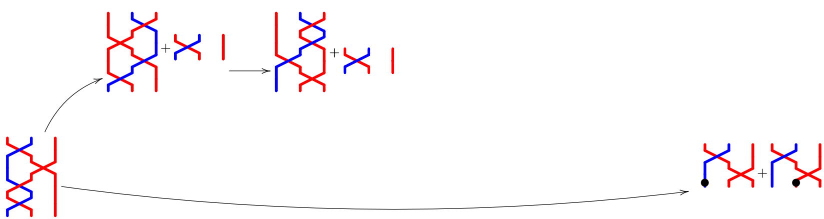
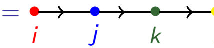


- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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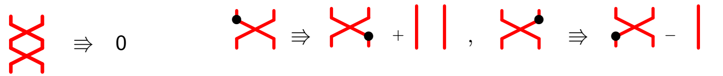

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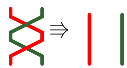

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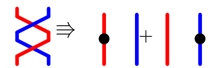

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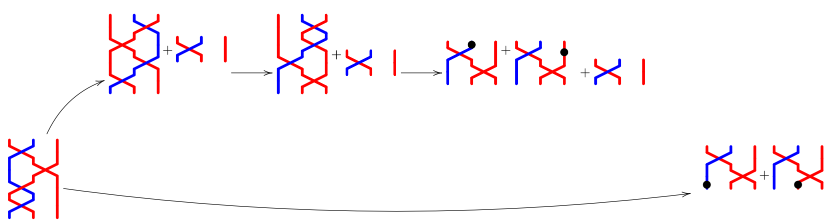

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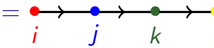
- Relations to realize the algebras $R(\mathcal{V})$ as 2Hom-spaces of a linear 2-category: ($\Gamma =$ )

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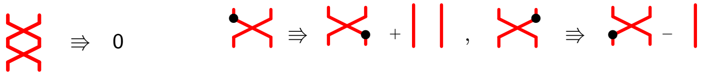

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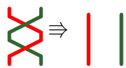

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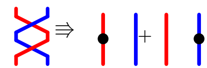

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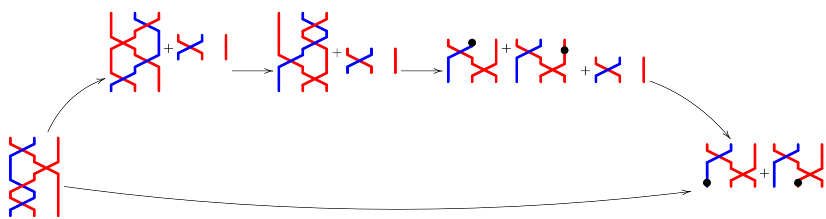

III. Confluence modulo in the KLR 2-category

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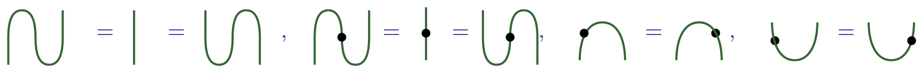

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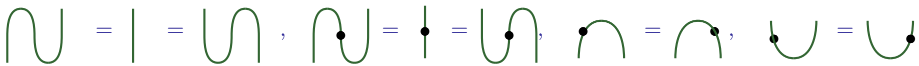

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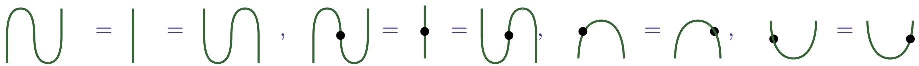

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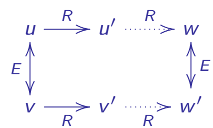

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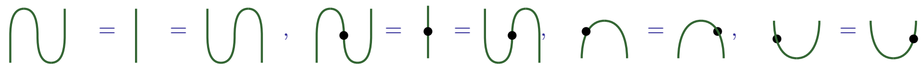

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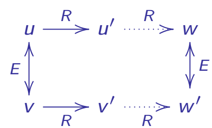

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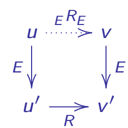

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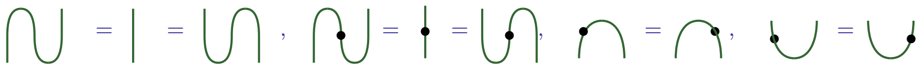

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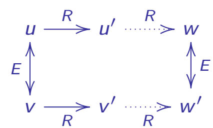

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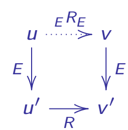

  - Rewriting system modulo: (R, E, S) such that $R \subseteq S \subseteq {}_ER_E$, **Jouannaud-Kirchner '84**.

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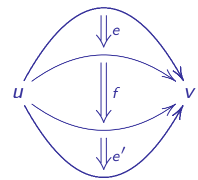

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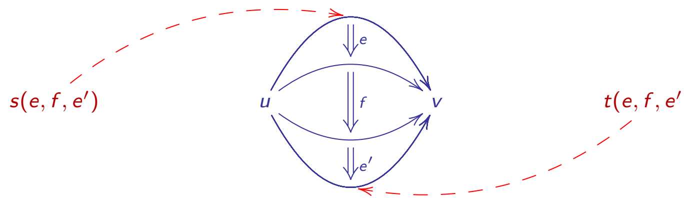

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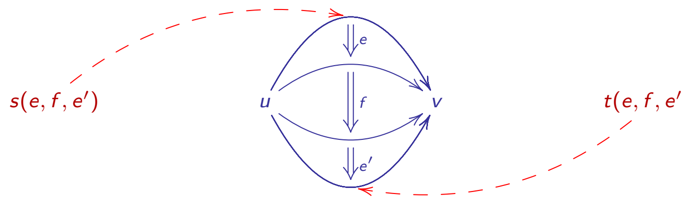

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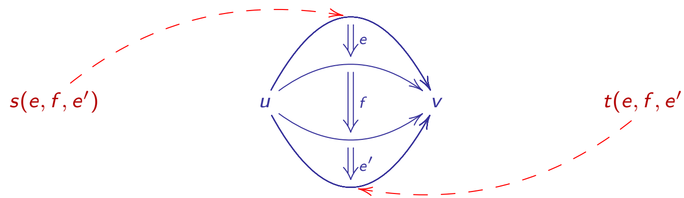

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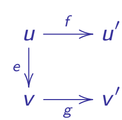

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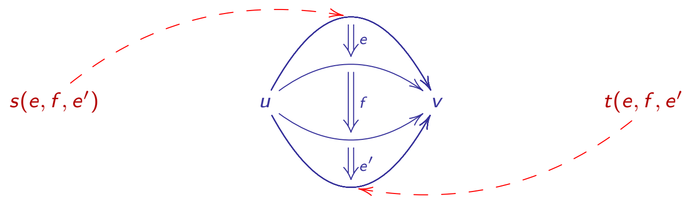

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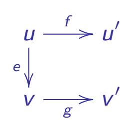

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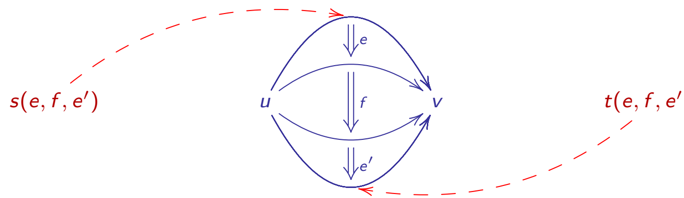

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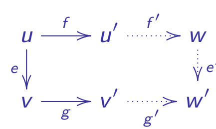

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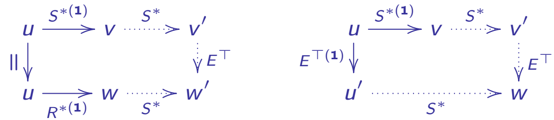
Linear bases from confluence modulo

- **Theorem [D' 19] (Linear critical branching lemma modulo):** If ${}_E R_E$ is terminating, S is locally confluent modulo E iff the critical branchings of the form



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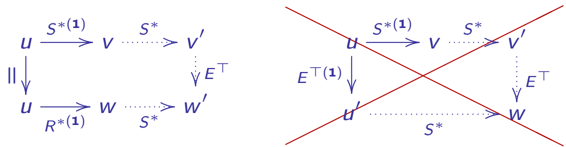

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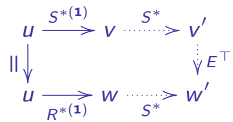

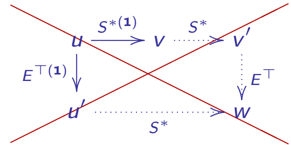

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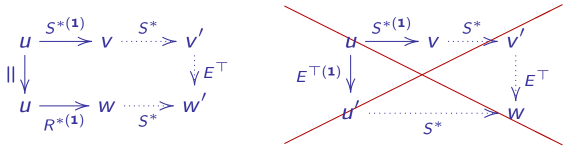

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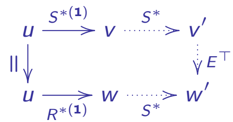

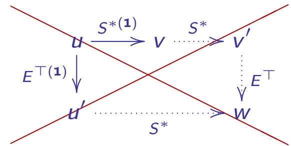

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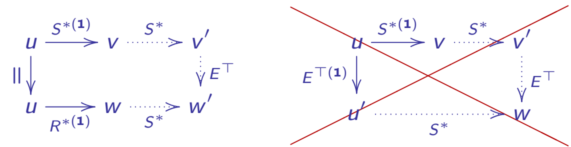

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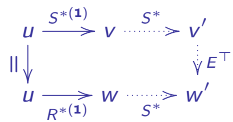

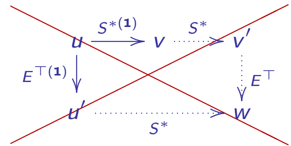

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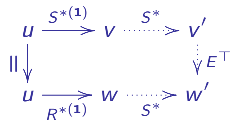

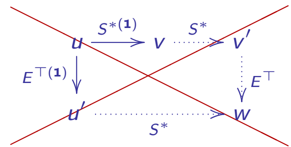

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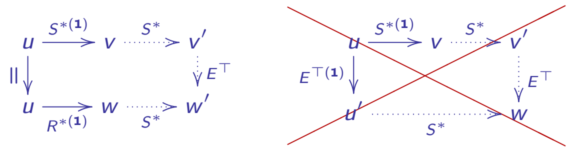

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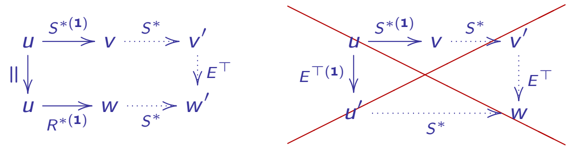

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- Reduce u into $\sum \widetilde{u}_i$, where $\widetilde{u}_i$ is a fixed monomial in quasi normal-form, that is for every 3-cell $v \Rrightarrow \widetilde{u}_i$, there exists a 3-cell $\widetilde{u}_i \Rrightarrow v$.

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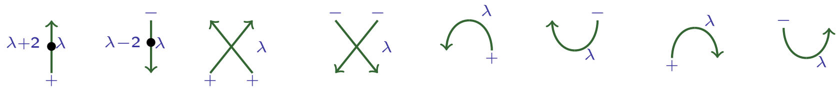

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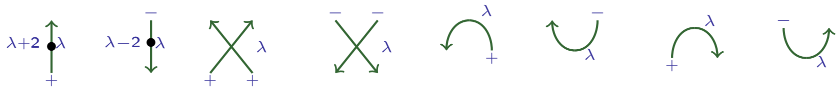

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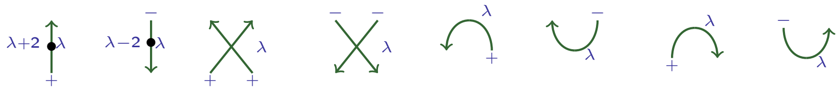

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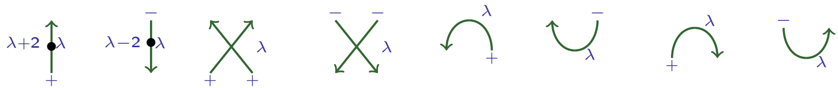

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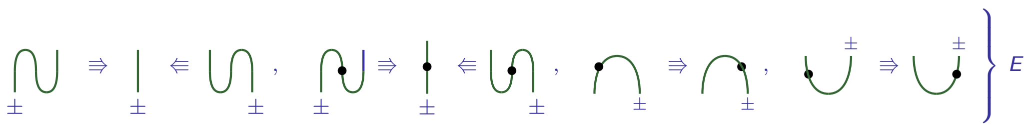

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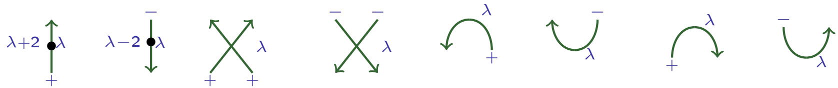

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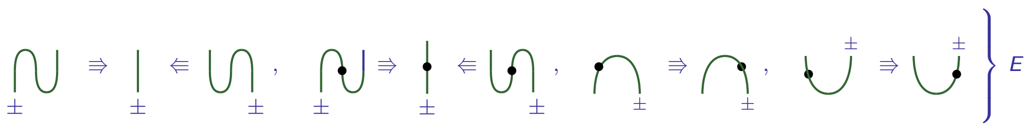

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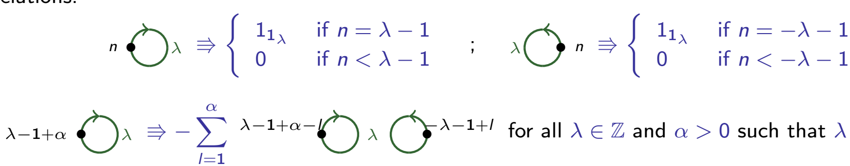

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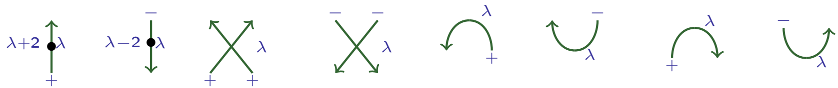

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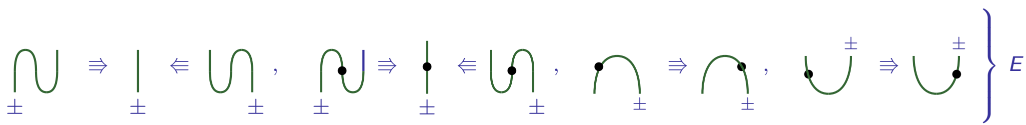

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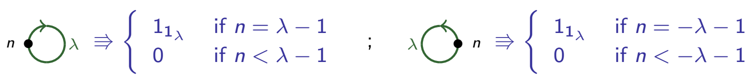

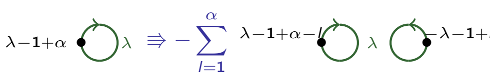

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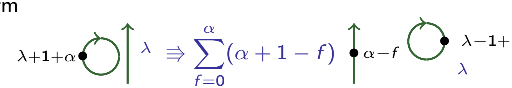

for any orientations of the bubbles and of the strand.

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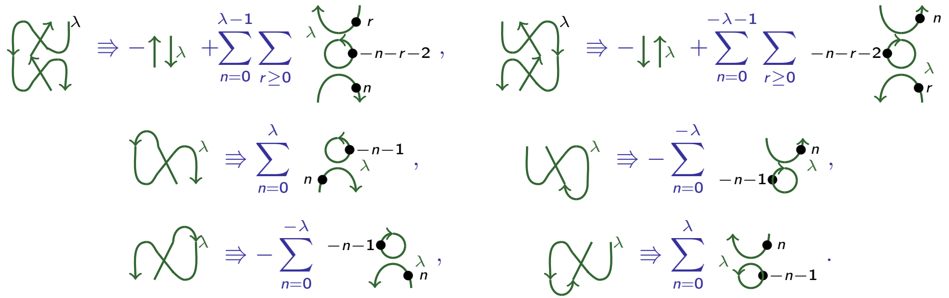

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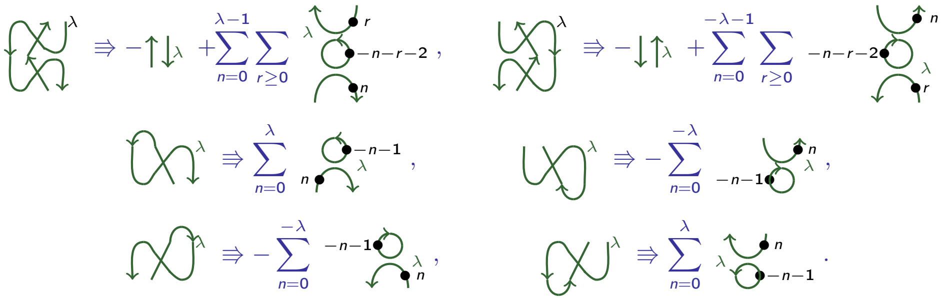

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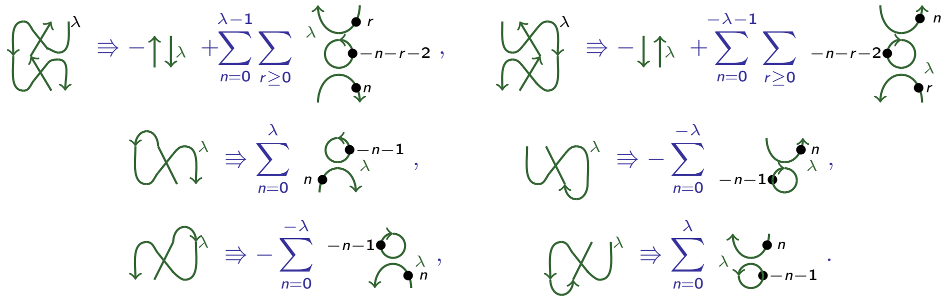

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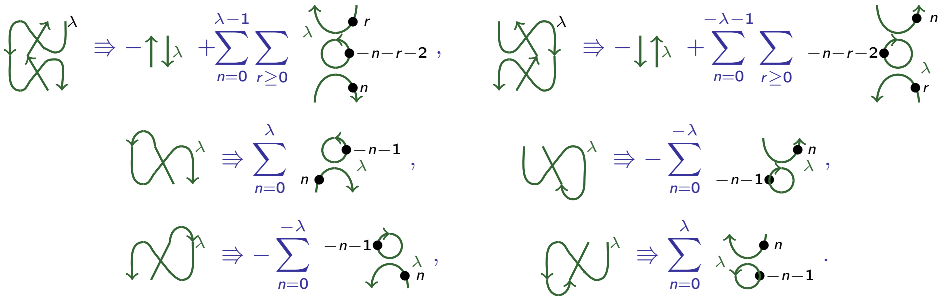

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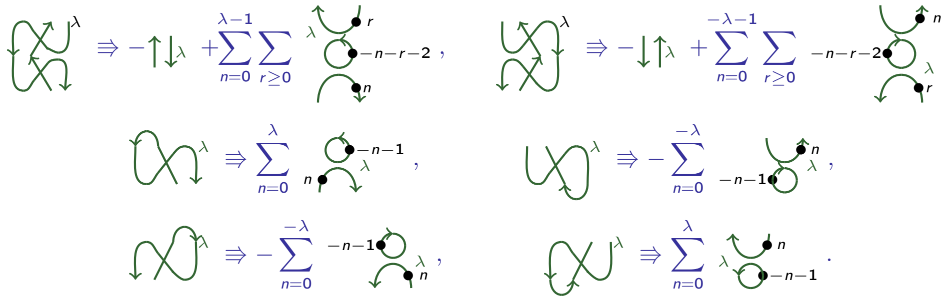

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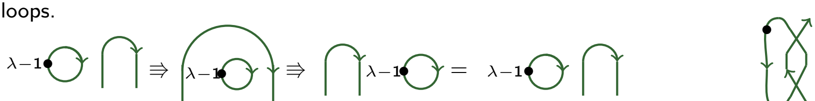

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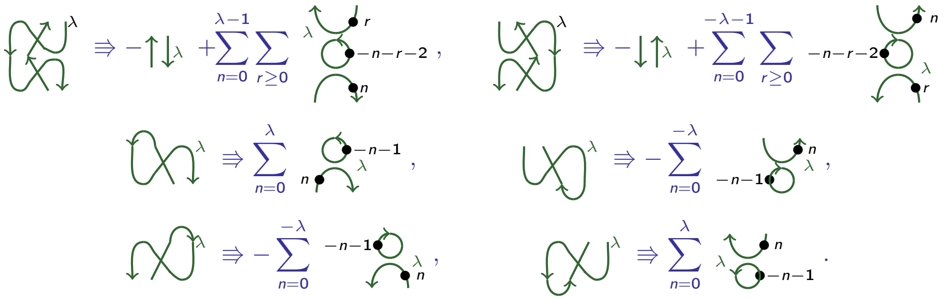

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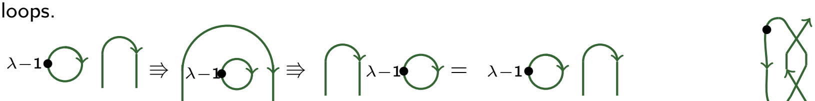

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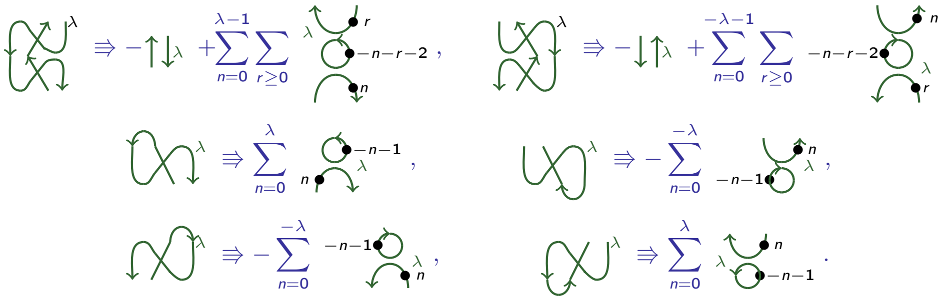

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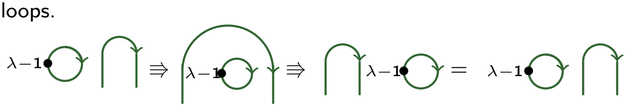

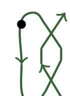

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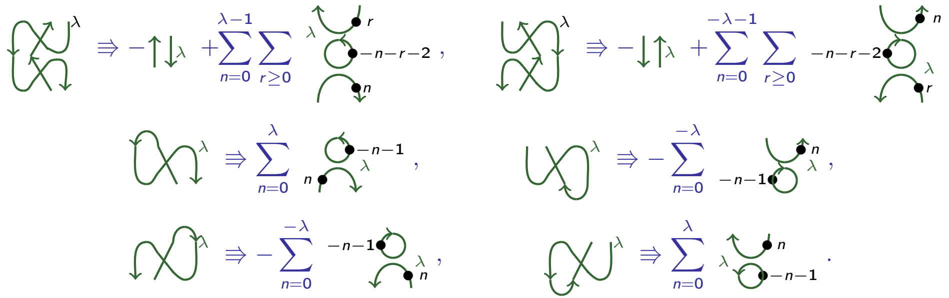
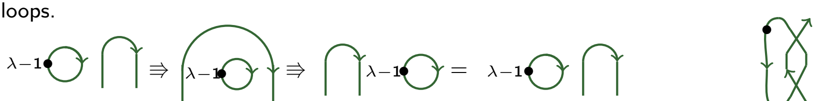
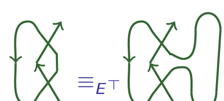


- Split this linear $(3,2)$-polygraph into E made of isotopy 3-cells and R containing the remaining relations.
- **Theorem [D' 19]:**
  - R is terminating without bubble slide 3-cells.
  - R and ${}_E R$ are quasi-terminating.
  - ${}_E R$ is confluent modulo E.
- **Sketch of the proof:**
  - Successive derivations to reduce the set of 3-cells that we need to prove terminating.
  - Procedure to reduce any 2-cell into a linear combination of 2-cells on which we can only derive bubble slide loops and indexed isotopy loops.



  - Exhaustive study of all the critical branchings modulo.



Non degeneracy of Khovanov-Lauda's diagrammatic calculus

- **Corollary:** A fixed set of quasi-normal forms containing diagrams with source p and target q in normal form with respect to E and having:
  - no loops,
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  - dots placed at the bottom of through strands and to the rightmost interval of arcs,
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- This holds for any simply-laced Kac-Moody algebra $\mathfrak{g}$.

IV. Conclusion and perspectives

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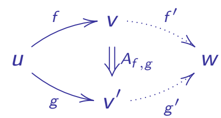

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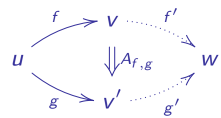

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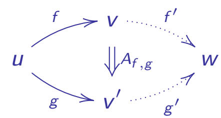

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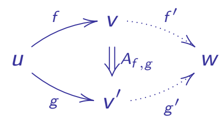

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- **Objective:** extend these constructions in higher dimensions.

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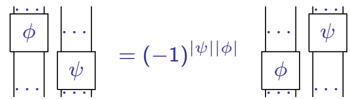

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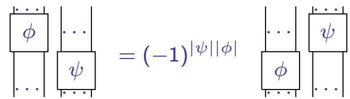

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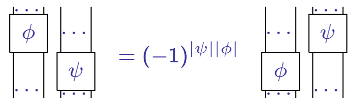

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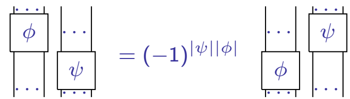

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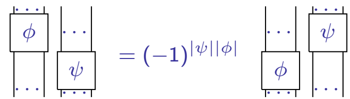

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- **Long-term project:** Implement computational tools to analyse confluence of diagrammatic presentations.

Thank you for your attention.