Coherence modulo relations

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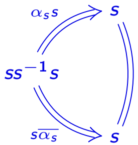
- Seeing a group $G = \langle X \mid R \rangle$ as a monoid $M = \langle X \coprod \overline{X} \mid R \cup \{xx^- \overset{\alpha_x}{\Rrightarrow} 1, x^- x \overset{\overline{\alpha_x}}{\Rrightarrow} 1\}_{x \in X}$, the confluence diagram



  is an artefact induced by the algebraic structure and should not be considered as a syzygy.

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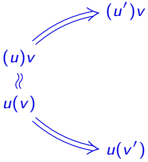

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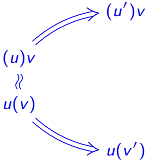

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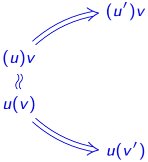

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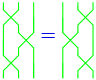

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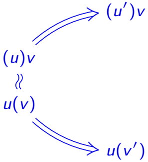

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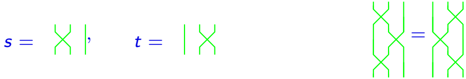
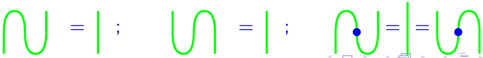


  - Rewriting in higher dimensional diagrammatic algebras, modulo the axioms of vector spaces and isotopies diagrams given by relations of the form



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- In this work, we use Huet's approach and generalize Squier's theorem for SRS to a coherence result modulo.

Plan of this talk

I. Confluence modulo

II. Coherence from confluence modulo

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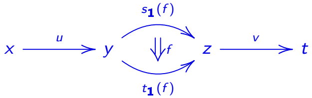
  - Each 2-cell f of R^* can be decomposed into a sequence $f = f_1 \star_1 f_2 \star_1 \ldots \star_1 f_k$, where each f_i is a 2-cell corresponding to a rewriting step of the form:



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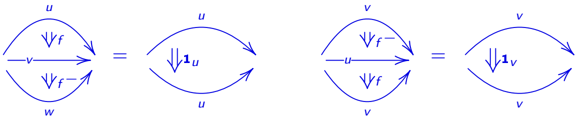

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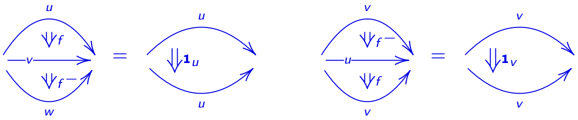

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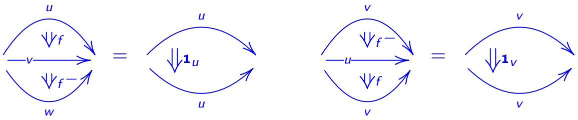

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- A 2-cell $u \Rightarrow v$ in $R^{\top}$ is given by a zigzag rewriting sequence of 2-cells of R^*:

$$f_1 \star g_1^{-1} \star_1 \cdots \star_1 f_n \star_1 g_n^{-1}$$

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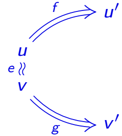

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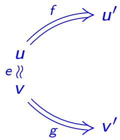

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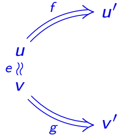

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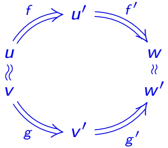
- A branching (f, g) is confluent modulo E if there exists 2-cells f' and g' in R^* such that



- R is confluent modulo E if all of its branchings are confluent modulo E.

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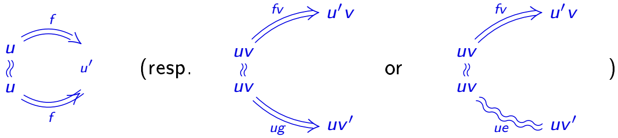

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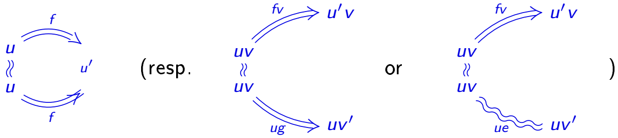

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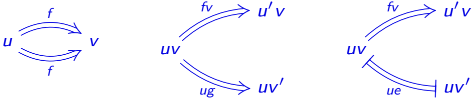

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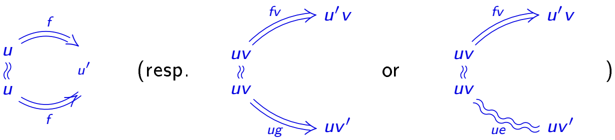

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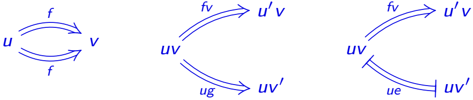

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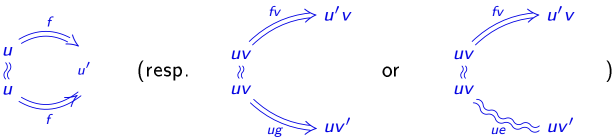

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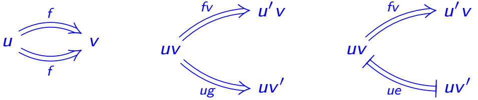

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- A critical branching modulo E is an overlapping local branching that is minimal for the

Newman and critical pair lemmas modulo

- **Theorem.** [Huet '80]
  If the SRS R_E containing rules of the form $u \Rightarrow v$ if there exists v' in X^* such that $v \approx_E v'$ and $u \Rightarrow v'$ is in R is terminating, then

  (R confluent modulo E) iff (Overlappings of R are confluent modulo E)

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- Proof based on Noetherian induction principle applied to an auxiliary SRS on $X \times X$ and the property:

$$\mathcal{P}(x,y): \quad x \approx_E y \;\Rightarrow \Big(\forall\, x', y' \mid x \overset{*}{\Rightarrow}_R x' \;\&\; y \overset{*}{\Rightarrow}_R y' \;\; \text{implies} \;\; x' \overset{E}{\vee} y' \Big)$$

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Newman and critical pair lemmas modulo

- **Theorem.** [Huet '80]
  If the SRS R_E containing rules of the form $u \Rightarrow v$ if there exists v' in X^* such that $v \approx_E v'$ and $u \Rightarrow v'$ is in R is terminating, then

$$(R \text{ confluent modulo } E) \quad \text{iff} \quad (\text{Overlappings of } R \text{ are confluent modulo } E)$$

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- **Theorem.** [Huet '80]
  R is locally confluent modulo a SRS E if and only if any critical branching of R modulo E is confluent modulo E.

II. Coherence from confluence modulo

2-Spheres modulo E

- A 2-sphere modulo E in $R^\top$ is a pair (f, g) of both non trivial 2-cells in $R^\top$ which are parallel modulo E, that is $s_1(f) \approx_E s_1(g)$ and $t_1(f) \approx_E t_1(g)$. We denote by $\mathrm{Sph}_E(R)$ the set of 2-spheres modulo E in $R^\top$.

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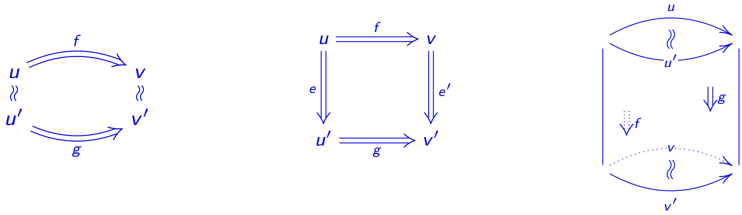

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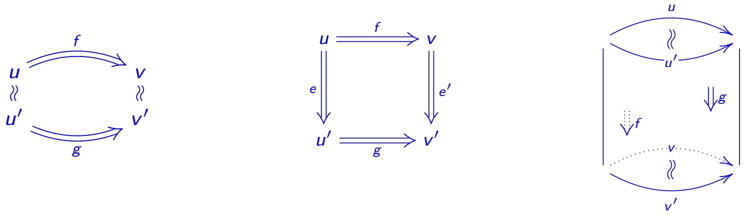

- A cellular extension of $R^{\top}$ modulo E is a set Γ equipped with a map $\gamma : \Gamma \longrightarrow \mathrm{Sph}_E(R)$, whose elements are called 3-cells modulo E.

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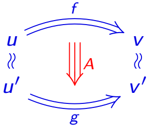

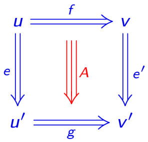

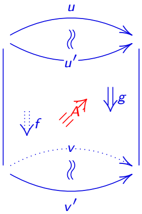

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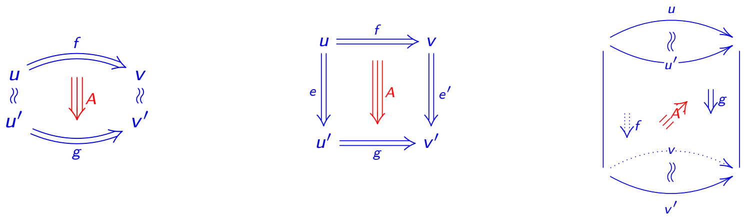

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- We say that Γ is coherent if the map γ is surjective, that is if each 2-sphere modulo E can be filled with a 3-cell of Γ.

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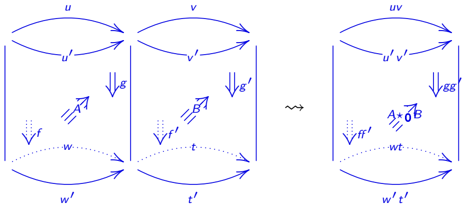

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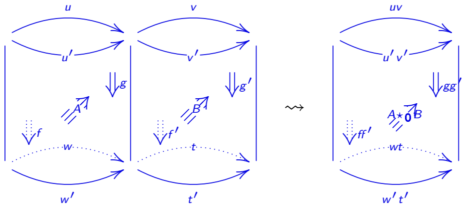

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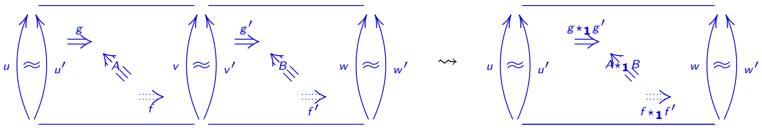

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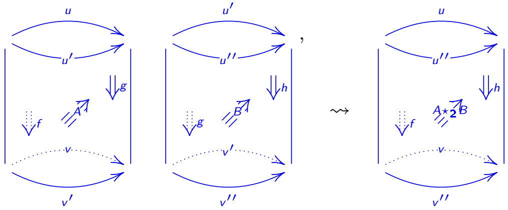

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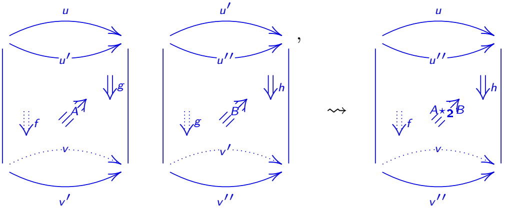

- We denote by $\mathcal{C}(\Gamma)$ the closure of Γ with respect to compositions $\star_0$, $\star_1$ and $\star_2$ of 3-cells of Γ and their formal inverses A^{-1} for $A \in \Gamma$ quotiented by:

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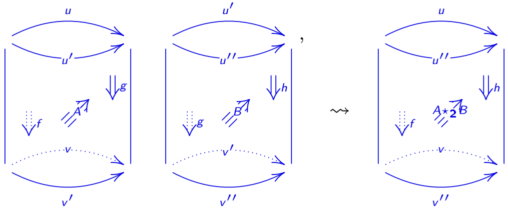

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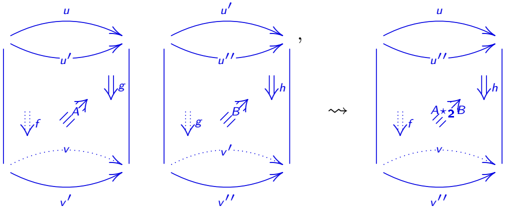

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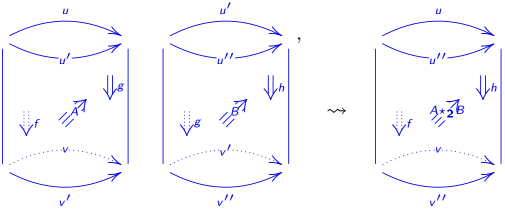

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- We say that Γ is an acyclic extension modulo E of $R^\top$ if $\mathcal{C}(\Gamma)$ is a coherent extension modulo E of $R^\top$.

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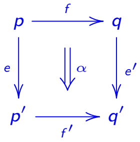
  - 1-cells of $\mathcal{C}_1$ are 2-cells in $\mathcal{C}$, pictured by  with p, p', q, q' 0-cells of $\mathcal{C}_0$.

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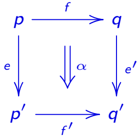

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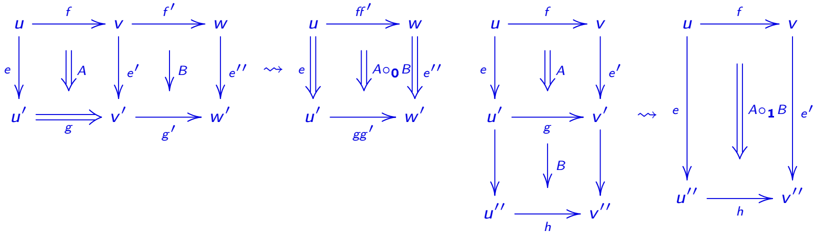

corresponding to $\star_1$ and $\star_2$-compositions in $\mathcal{C}(\Gamma)$.

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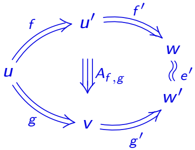

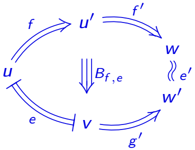

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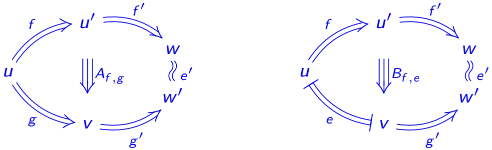

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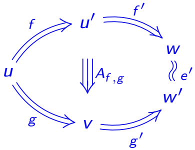

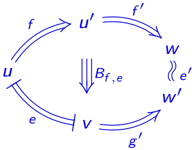

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- The proof of this theorem is separated into 5 steps.

Proof: Step 1

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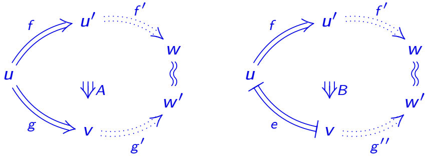

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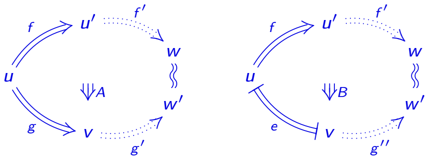

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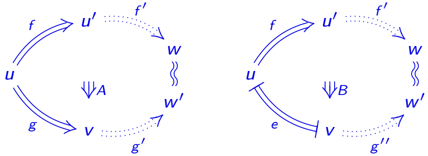

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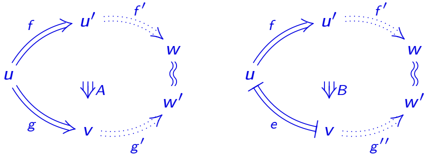

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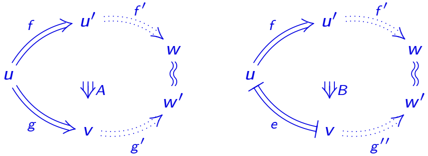

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  - If (f, g) (resp. (f, e)) is an overlapping that is not critical, we have $(f, g) = (uhv, ukv)$ (resp. $(f, e) = (uhv, ue'v)$) for some u, v in X^* such that both (h, k) and (h, e') are critical.

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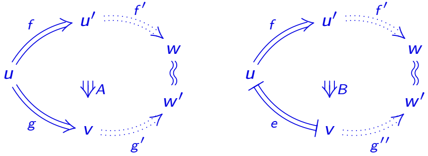

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  - We consider the 3-cells $A' : f \star_1 f' \Rrightarrow_E g \star_1 g'$ and $B' : f \star_1 f' \Rrightarrow_E e \star_1 g''$ corresponding respectively to the critical branchings (h, k) and (h, e'). We conclude by setting

$$f' = uh'v \quad g' = uk'v \quad g'' = ue'v \quad A' = u \star_0 A' \star_0 v \quad B = u \star_0 B' \star_0 v.$$

Proof: Step 2

- For any 2-cells $f : x \Rightarrow y$ and $g : x \Rightarrow z$ of R^* with $y \approx_E z$, there exists a 3-cell modulo E from f to g in $\mathcal{C}(\mathcal{S}(R, E))$:

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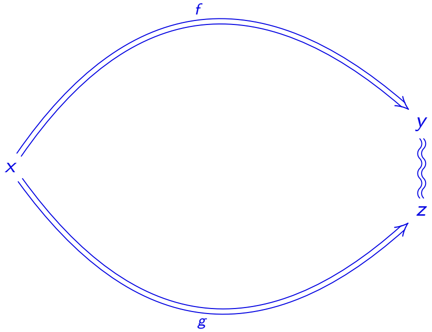

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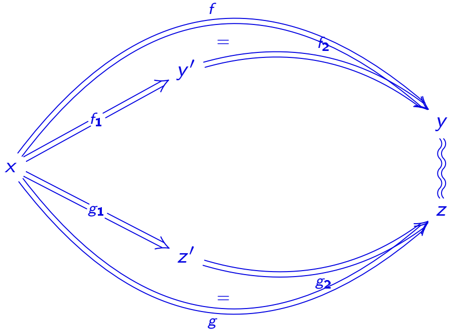

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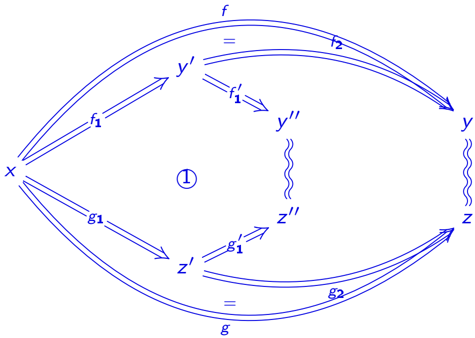

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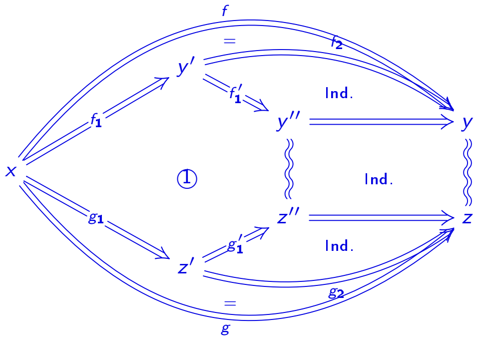

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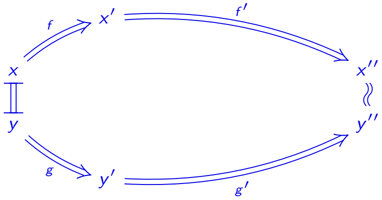

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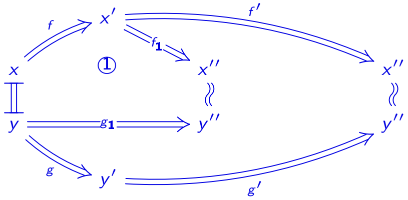

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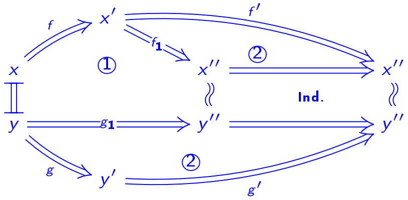

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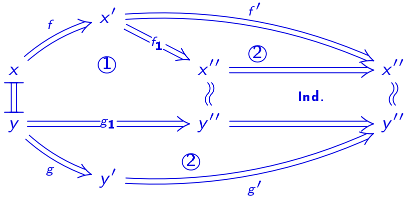
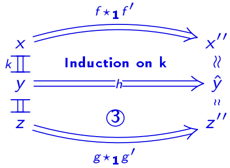


  - Suppose the result proved for $\ell(e) = k > 1$ and let us prove the result for $\ell(e) = k + 1$.



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- For any 2-cells $f : x \Rightarrow \hat{x}$ and $g : y \Rightarrow \hat{y}$ with $x \overset{e}{\approx}_E y$, there exists a 3-cell $A : f \Rrightarrow_E g$ modulo E in $\mathcal{C}(\mathcal{S}(R, E))$.

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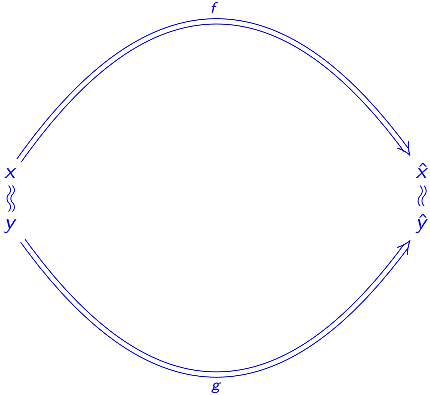

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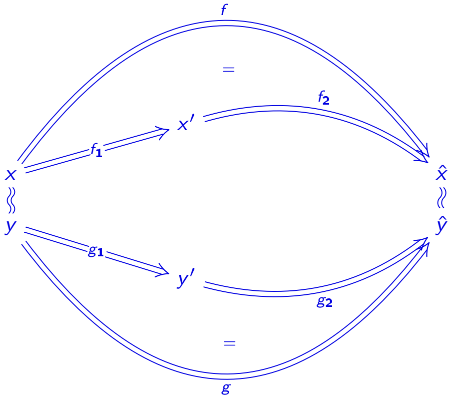

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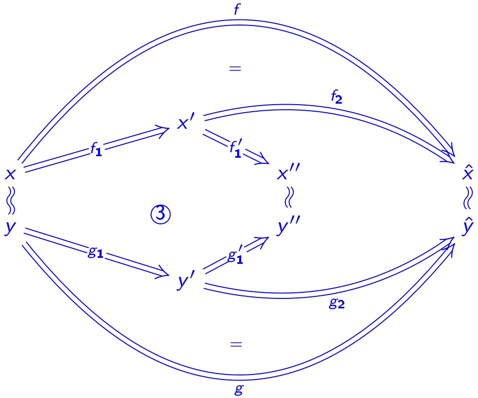

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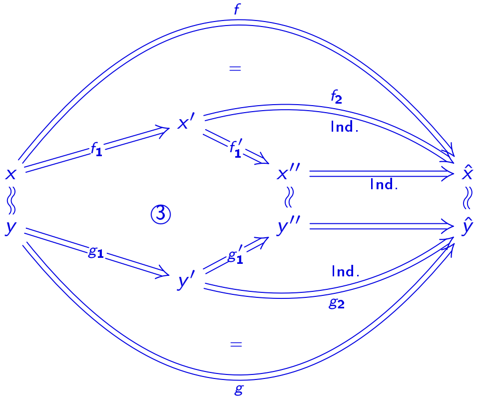

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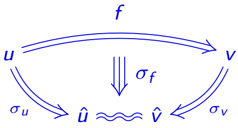

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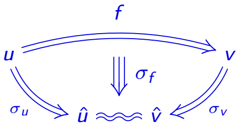

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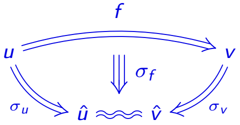

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$$u \overset{f_1}{\Longrightarrow} v_1 \overset{g_1}{\Longleftarrow} u_2 \Rightarrow (\cdots) \Leftarrow u_n \overset{f_n}{\Longrightarrow} v_n \overset{g_n}{\Longleftarrow} v$$

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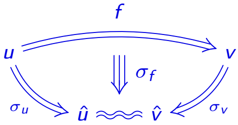

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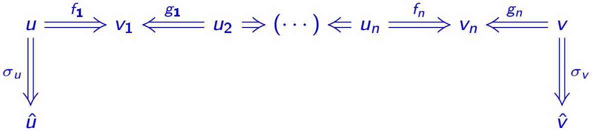

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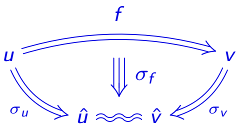

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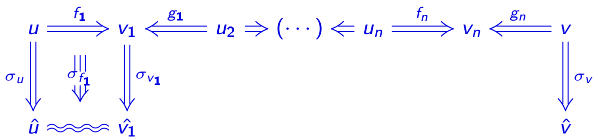

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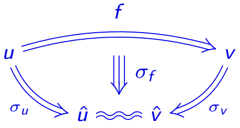

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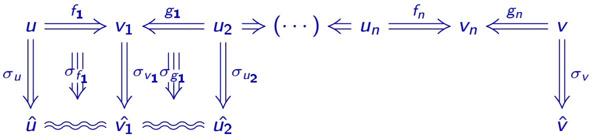

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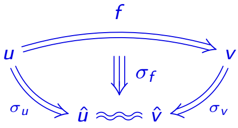

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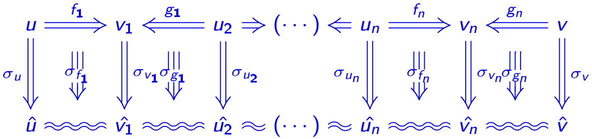

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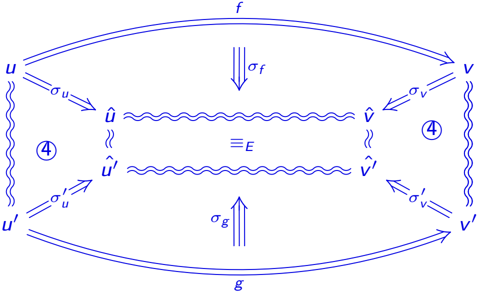

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  - Describe this completion in terms of critical pairs.
- The main application is to obtain homotopical completions modulo, and in particular constructions of coherent presentations for
  - groups;
  - diagrammatic algebras.