Termination in linear $(2, 2)$-categories with braidings and duals

Dupont Benjamin

Higher Dimensional Rewriting and Algebra

Oxford, 7 July 2018

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  - The two essential properties to study are termination and confluence.

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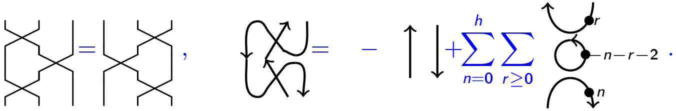

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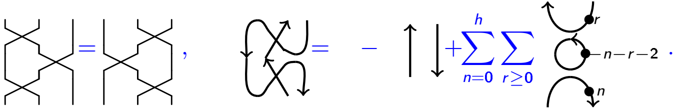

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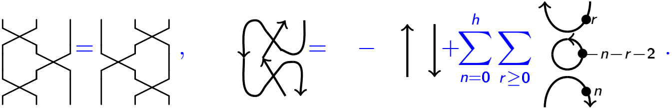

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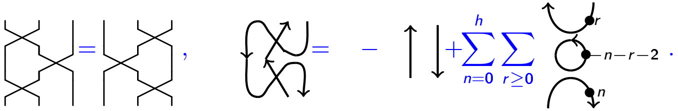

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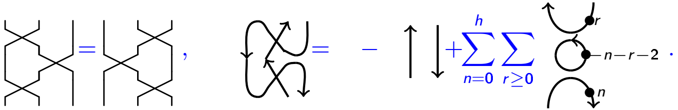

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- In this work, we construct heuristics to prove termination of some diagrammatic rewriting systems.

- **Main problem:** A diagrammatic rewriting system does not always admit a monomial (total and well-founded) termination order.

- We will define termination orders similar to monomial orders, counting the generators in the diagrams, stable by contexts and well-founded, but that are not required to be total.

Plan of this talk

I. Linear $(2,2)$-categories, braidings and duals

II. Decreasing order operators

III. Termination heuristics in particular linear $(2,2)$ categories

IV. Illustration on the diagrammatic rewriting system $\mathcal{KLR}$

I. Linear $(2, 2)$-categories, braidings and duals

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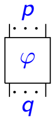

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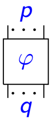
- The 2-cells in $\mathcal{C}$ between 1-cells p and q are represented by diagrams 
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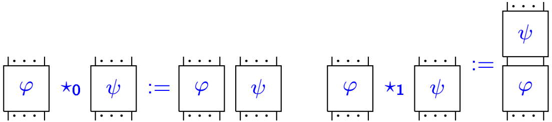

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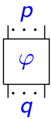

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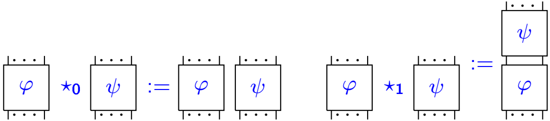
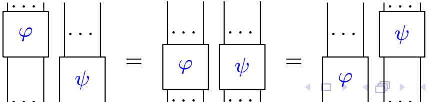


  - modulo the exchange law of $\mathcal{C}$, diagrammatically depicted as



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- We consider diagrammatic algebras interpreted as linear $(2,2)$-categories, admitting a diagrammatic presentation by generators and relations with:
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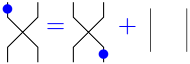

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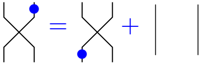

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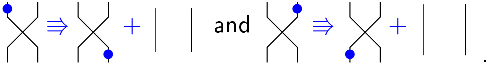

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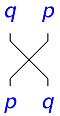
(or 

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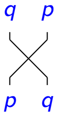
(or 

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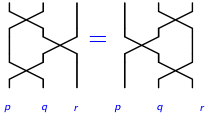

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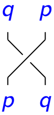
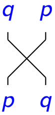
- A braiding on a linear $(2,2)$-category $\mathcal{C}$ is a family of 2-cells $\sigma_{p,q} : p \star_0 q \to q \star_0 p$ for any p, q in $\mathcal{C}_1^g$, represented by 

(or 

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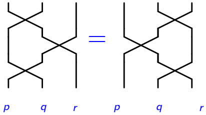

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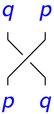

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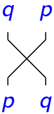

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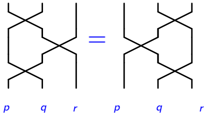

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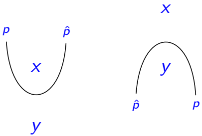

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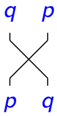
(or 

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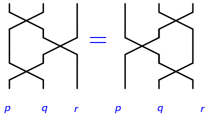

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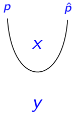

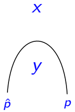

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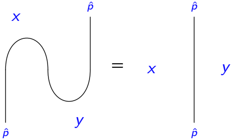

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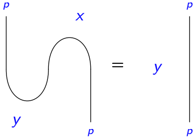
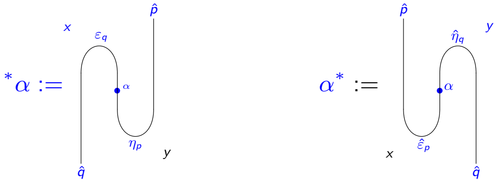


Cyclic 2-cells

- Given a pair of 1-cells $p, q : x \to y$ in $\mathcal{C}$ with chosen biadjoints $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$, then for any 2-cell $\alpha : p \Rightarrow q$, we construct two duals ${}^*\alpha$ and $\alpha* : \hat{q} \Rightarrow \hat{p}$ as follows:



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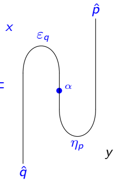
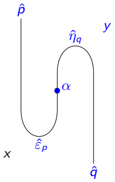
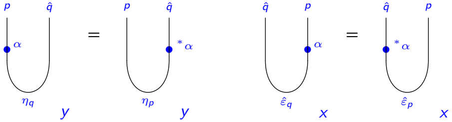
${}^*\alpha :=$ 

$\alpha^* :=$ 

- A 2-cell $\alpha : p \Rightarrow q$ is said cyclic if the equation ${}^*\alpha = \alpha^*$ is satisfied, or either of the equivalent conditions ${}^{**}\alpha = \alpha$ or $\alpha^{**} = \alpha$ are satisfied, yielding relations of the form



(1)

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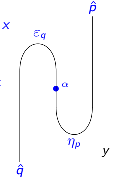
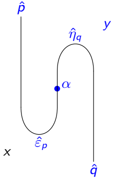
$^*\alpha :=$ 

$\alpha^* :=$ 

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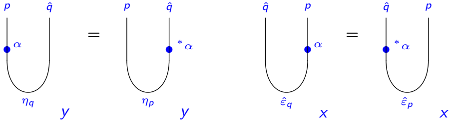

(1)

- A linear $(2, 2)$-category $\mathcal{C}$ in which any 2-cell α is cyclic is called a pivotal category.

II. Decreasing order operators

Decreasing order operators

- We want to construct a termination order on the set of 2-cells of a DRS Σ:

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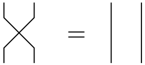
- Example.  $\rightsquigarrow$ number of crossings.
- Given a DRS Σ, one defines a decreasing order operator (DOO) for Σ as a family of functions $\Phi_{p,q} : \Sigma_2(p,q) \to \mathbb{N}^{m(p,q)} \times \mathbb{Z}$ indexed by 1-cells p and q, satisfying:

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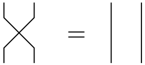
- Example.  $\rightsquigarrow$ number of crossings.
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  - For any 3-cell $\alpha : D_1 \Rrightarrow D_2$ with D_1,D_2 in $\Sigma_2(p,q)$, the function $\Phi_{p,q}$ satisfy

$$\Phi_{p,q}(D_1) > \Phi_{p,q}(D')$$

    where $>$ is the lexicographic order on $\mathbb{N}^{m(p,q)} \times \mathbb{Z}$ and D' is a monomial in D_2. We denote this by $D_1 >_{\text{lex}} D_2$.

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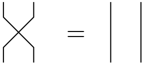

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  - The $\Phi_{p,q}$ are stable by context: for any D_1 and D_2 in $\Sigma_2(p,q)$ and any context C of Σ, if $D_1 >_{\text{lex}} D_2$, then $C[D_1] >_{\text{lex}} C[D_2]$.

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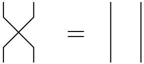
- Example.  $\rightsquigarrow$ number of crossings.
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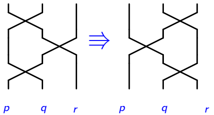
  - The $\Phi_{p,q}$ are stable by exchange law.

III. Termination heuristics in particular linear $(2,2)$ categories

Termination with braid relations

- Let **Crs** be the DRS having: only one 0-cell, a set of generating 1-cells $\mathbf{Crs}_1$, for 2-cells the braidings $\sigma_{p,q}$ for each p and q in $\mathbf{Crs}_1$, and 3-cells as follows:



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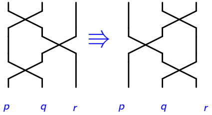

- **Crs** is terminating by the DOO $\Phi_{p,q}$ counting the number $\mathrm{yb}(D)$ of occurences of 2-cells $\sigma_{p,q} \star_0 \mathrm{id}_r$ in a diagram D, for p,q and r in $\mathbf{Crs}_1$.

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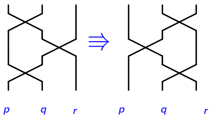

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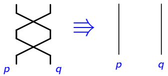
- Let $\mathbf{Crs}'$ be the DRS defined by

$$\mathbf{Crs}' = \mathbf{Crs} \cup \left\{ \quad \right\}$$



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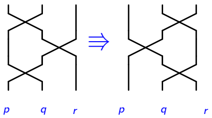

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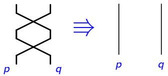

$$\left. \right\}$$

- We add as first component to the $\Phi_{p,q}$ defined for $\mathbf{Crs}$ a component counting the number of crossings of the diagrams.

Termination with braid relations and additional 2-cells

- Let $\mathbf{Crs}^{\mathsf{add}}$ be a DRS defined by

$$\mathbf{Crs}^{\mathsf{add}} = \left(\mathbf{Crs}_0', \mathbf{Crs}_1', \mathbf{Crs}_2' \cup \left\{ \begin{array}{c} q \\ \bullet\, \alpha \\ q \end{array} \quad \text{for } q \text{ in } \mathbf{Crs}_1' \right\}, \mathbf{Crs}_3' \right)$$

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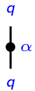
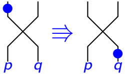


$$\text{for } q \text{ in } \mathbf{Crs}_1' \Big\}, \mathbf{Crs}_3'\Big)$$

- Assume that $\mathbf{Crs}^{\mathrm{add}}$ admits a 3-cell of the following form



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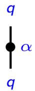

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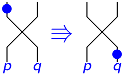

$$\Rrightarrow \quad + \text{lower terms for the previous DOO} \; .$$

- We define a new DOO as follows: for $p, q \in \mathrm{Crs}_1'$, we set $m := \max(\ell(p), \ell(q))$. We add to $\Phi p, q$ the components $(c_k(D))_{1 \leq k \leq m}$ defined by:

Termination with braid relations and additional 2-cells

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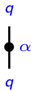

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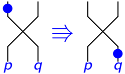

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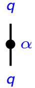

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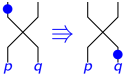

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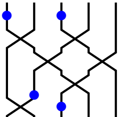
- Example. For

$$D =$$



Example: the nil Hecke algebra

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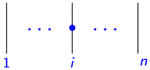
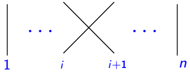
  - generators $\xi_i =$ 

for $1 \leq i \leq n$ and $\partial_i =$ 

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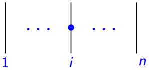
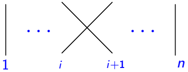
  - generators $\xi_i =$ 

for $1 \leq i \leq n$ and $\partial_i =$ 

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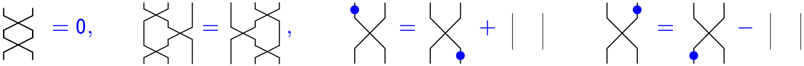

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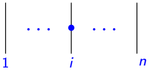
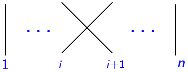
  - generators $\xi_i =$ 

 for $1 \leq i \leq n$ and $\partial_i =$ 

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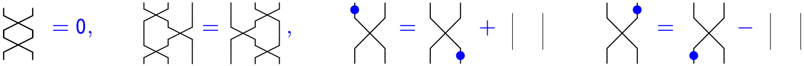

- $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$ form a linear $(2,2)$-category with only one 0-cell, the 1-cells are permutations and 2-cells are braiding diagrams.

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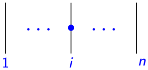
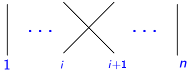
  - generators $\xi_i =$ 

for $1 \leq i \leq n$ and $\partial_i =$ 

for $1 \leq i < n$;
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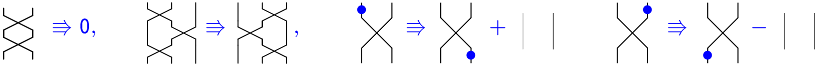

- $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$ form a linear $(2,2)$-category with only one 0-cell, the 1-cells are permutations and 2-cells are braiding diagrams.

- Let Σ be a DRS presenting $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$ with relations oriented as above.

Example: the nil Hecke algebra

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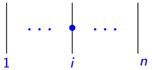
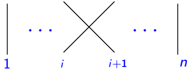
  - generators $\xi_i =$ 

for $1 \leq i \leq n$ and $\partial_i =$ 

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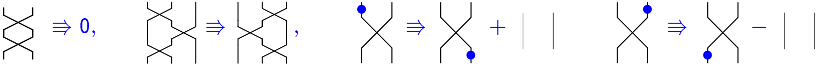

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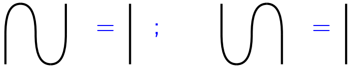
- We prove that Σ is terminating using the following DOO: for a given diagram D in $\mathcal{NH}_n^0(\sigma, \tau)$,

$$\Phi_{\sigma,\tau}(D) = (c(D), \mathrm{yb}(D), c_1(D), \ldots, c_n(D)).$$

Termination with adjunctions

- Let $\mathcal{C}$ be a linear $(2,2)$-category whose 1-cells are equipped with biadjunctions, yielding isotopy relations of the form



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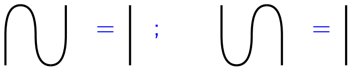
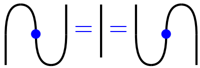


- If there is an additional 2-cell α which is cyclic wrt biadjunction $p \vdash q \vdash p$, we have to impose some new relations of the form:



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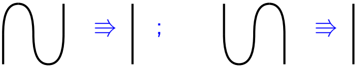

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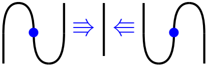
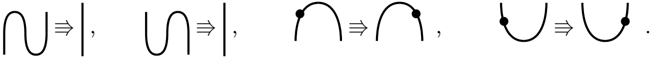


- The DRS given by these orientations is not confluent: the first Knuth-Bendix step imposes to add the following relations:

Prototypical example: the 3-polygraph of pearls

- Let **Pearl** be the DRS defined by:
  - only one 0-cell $*$;
  - only one 1-cell p;
  - generating 2-cells: 
  - the following 3-cells:



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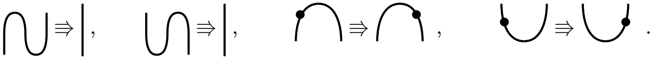

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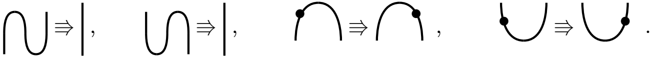

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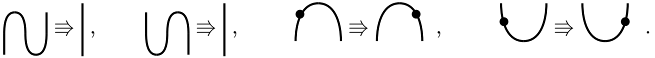

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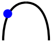
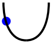
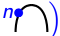
  - $\text{l-dot}(D)$ corresponds to the number of positively left-dotted caps and cups, that is the number of elements  and  (with at least one $\bullet$) appearing in D with the convention

$$\text{l-dot}\left(\right.$$ 

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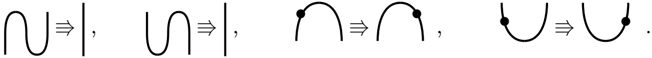

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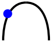
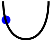
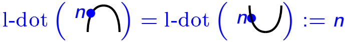
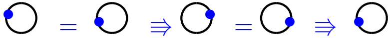
  - $\text{l-dot}(D)$ corresponds to the number of positively left-dotted caps and cups, that is the number of elements  and  (with at least one $\bullet$) appearing in D with the convention



- Adding a $\star_0$ and $\star_1$-context to D, we add a constant number of cups and caps, and $\text{l-dot}(D)$ can not increase since a dot cannot move from right of a cap/cup to its left even by adding a context.

Termination or quasi-termination ?

- If we choose different orientation for the dot move relations, we create rewriting cycles



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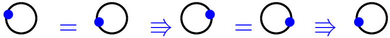

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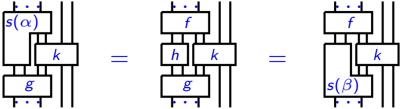
- In a DRS Σ, there are indexed critical branchings of the following form:



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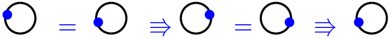

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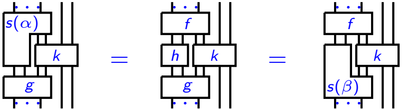

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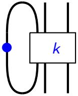

Example. If Σ contains braiding 2-cells $\sigma_{p,q}$ for any p,q in Σ_1, there are infinitely many indexed critical branchings in Σ for each $n \in \mathbb{N}$.

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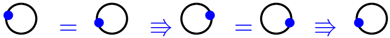

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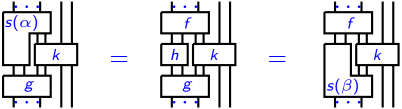

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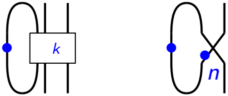

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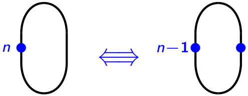
Quasi-termination

- A DRS Σ is quasi-terminating if for each rewriting sequence $(u_n)_{n\in\mathbb{N}}$ of 2-cells of Σ, it contains an infinite number of occurences of the same 2-cell.
- Let Σ be a DRS containing the following 3-cells:

Σ is not terminating, one wants to study its quasi-termination.

- A quasi-reduced monomial in Σ is a monomial on which we can only apply the rules



- We may prove that Σ is quasi-terminating by constructing a DOO on the sets $\text{Q-red}(\Sigma_2(p,q))$ of quasi-reduced monomials between two 1-cells p and q.
  - This DOO does not take into account the number of left-dotted cups and caps.
  - It ensures that there is no other obstruction to termination than the bubble cycles.

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- Proposition. There is a DRS Σ presenting $\mathcal{C}$ in which the relations are oriented in such a way that $\Phi_{p,q}(s(\alpha)) > \Phi_{p,q}(t(\alpha))$ for a DOO of Σ constructed as above and any 3-cell α, and thus Σ is terminating.

IV. Illustration on the linear $(3, 2)$-polygraph $\mathcal{KLR}$

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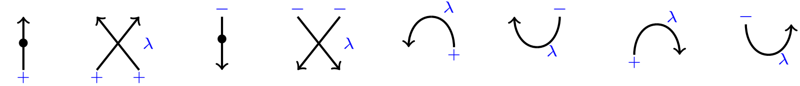

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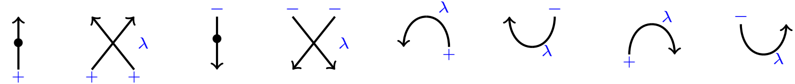

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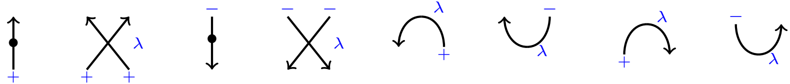

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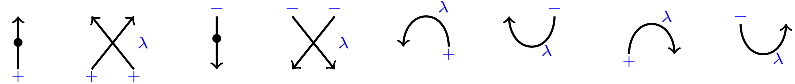

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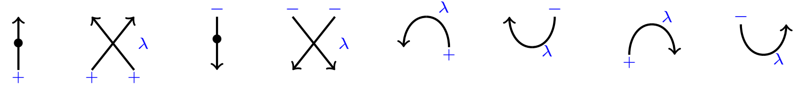

- The 3-cells in $\mathcal{KLR}_3$ are given by:
  - The 3-cells of the nilHecke algebras described previously.
  - The isotopy 3-cells;
  - Some bubble conditions 3-cells:

$$\Rrightarrow \begin{cases} 1_{1_\lambda} & \text{if } n = h-1 \\ 0 & \text{if } n < h-1 \end{cases}$$

$$\Rrightarrow \begin{cases} 1_{1_\lambda} & \text{if } n = -h-1 \\ 0 & \text{if } n < -h-1 \end{cases}$$

The linear $(3,2)$-polygraph $\mathcal{KLR}$

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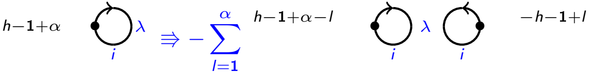
  - the infinite Grassmanniann relation: for any $\lambda \in X$ and $\alpha > 0$,



where h is a number given by the Kac-Moody algebra.

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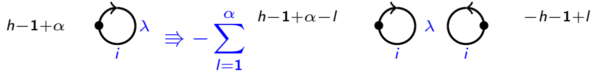

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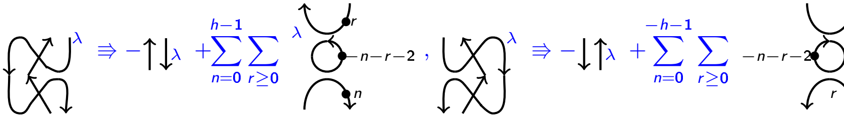
  - Some invertibility 3-cells:



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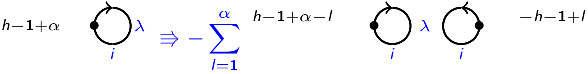

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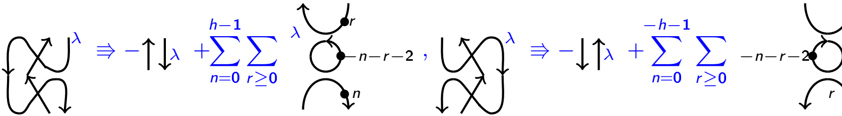
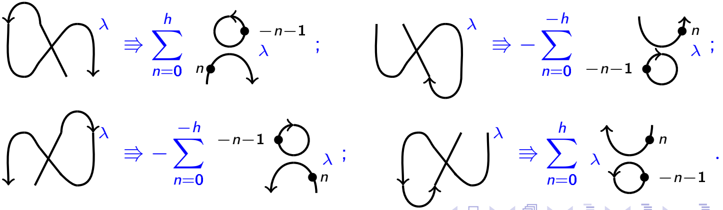


  - Some "$\mathfrak{sl}_2$" 3-cells:



The linear $(3,2)$-polygraph $\mathcal{KLR}$

- $\mathcal{KLR}$ is terminating using the following DOO:

$$\begin{array}{rcl} \Phi_{\varepsilon,\varepsilon'} : \mathcal{KLR}_2(\varepsilon,\varepsilon') & \to & \mathbb{N}^{m+4} \times \mathbb{Z} \\ D & \mapsto & (c(D), c_1(D), \dots, c_m(D), \text{ybg}(D), I(D), \text{l-dot}(D), \deg_b(D)) \end{array}$$

with:

- $c(D)$ is the number of crossings between strands in D;
- for $1 \leq k \leq m$, $c_k(D)$ is defined as above;
- $\text{ybg}(D)$ defined as above;
- $I(D)$ corresponds to the number of rightward caps and leftward cups that appear in D;
- $\text{l-dot}(D)$ corresponds to the number of positively leftward dotted caps and cups as described above.
-

$$\deg_b(D) := \begin{cases} \#\{\text{bubbles in } D\} + \sum_{\pi \text{ clockwise bubble in } D} \deg(\pi) & \text{if D is a diagram with} \\ 0 & \text{if D is a diagram witho} \\ -\infty & \text{if } D = 0. \end{cases}$$

Conclusion

- We presented heuristics to prove termination of some DRS presenting diagrammatic algebras coming from representation theory.
- The next question to study is confluence of these DRS.
  - The diagrammatic structure yield a combinatorial explosion for computation of critical pairs, as for instance isotopy relations.
  - Isotopy should not be considered as rewrite rules, but as equations we have to take into account when rewriting.
  - Develop a context of rewriting modulo isotopy, and obtain linear bases and coherence results in that setting.