

Termination in linear $(2,2)$ -categories with braidings and duals

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Higher Dimensional Rewriting and Algebra

Oxford, 7 July 2018

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- ▶ **Diagrammatic rewriting:** **3-dimensional** linear rewriting systems on diagrams
 - ▶ The two essential properties to study are **termination** and **confluence**.

Motivations: termination issues

- ▶ Consider a diagrammatic algebra A admitting relations of the form

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}, \quad \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = - \updownarrow + \sum_{n=0}^h \sum_{r \geq 0} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \cdot$$

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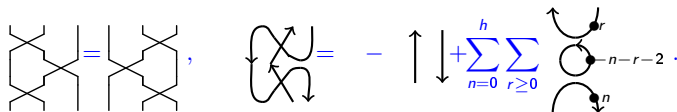
$$\text{Diagram 1} = \text{Diagram 2}, \quad \text{Diagram 3} = - \left(\uparrow \downarrow + \sum_{n=0}^h \sum_{r \geq 0} \text{Diagram 4} \right)$$

one naturally asks:

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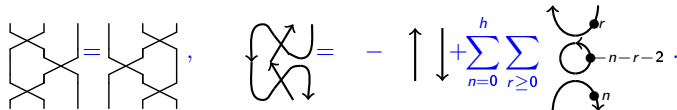
The image shows two equations. The first equation shows two diagrams of a crossing of two strands, one with a loop on the left strand, separated by an equals sign. The second equation shows a diagram of two strands crossing with loops on both, followed by an equals sign and a minus sign. To the right of the minus sign is a vertical double-headed arrow. To the right of the arrow is a plus sign followed by a double summation: $\sum_{n=0}^h \sum_{r \geq 0}$. To the right of the summations is a diagram of a circle with two strands passing through it, one from the top and one from the bottom, each with a loop. The top loop is labeled r and the bottom loop is labeled n . To the right of the circle is a dot with the label $n-r-2$.

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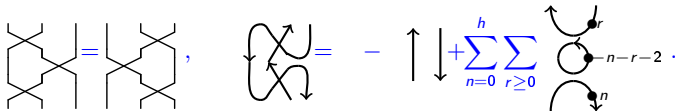
The image shows three diagrammatic relations. The first relation shows two diagrams of a crossing with a vertical line passing through it, separated by an equals sign. The second relation shows a diagram of two crossing lines with arrows, separated by an equals sign. The third relation shows a diagram of a circle with two dots and arrows, separated by an equals sign. To the right of the third relation is a mathematical expression: $+\sum_{n=0}^h \sum_{r \geq 0}$ followed by a diagram of a circle with two dots and arrows, and a dot with the label $n-r-2$.

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The image shows three diagrammatic equations. The first equation shows two diagrams of a crossing with a vertical line passing through it, separated by an equals sign and a comma. The second equation shows a diagram of two crossings with arrows, separated by an equals sign and a minus sign. The third equation shows a diagram of a circle with two dots and arrows, separated by an equals sign and a plus sign. The plus sign is followed by a double-headed vertical arrow, then a summation symbol with $n=0$ below it and $r \geq 0$ to its right, then another summation symbol with h above it and $r \geq 0$ to its right, and finally the diagram of the circle with two dots and arrows. The diagram of the circle has a dot at the top labeled r and a dot at the bottom labeled n , with a $-n-r-2$ next to it.

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- ▶ How should we orient the rules to obtain a terminating presentation of A ?
- ▶ In this work, we construct heuristics to prove termination of some diagrammatic rewriting systems.
- ▶ **Main problem:** A diagrammatic rewriting system does not always admit a monomial (total and well-founded) termination order.
- ▶ We will define termination orders similar to monomial orders, counting the generators in the diagrams, stable by contexts and well-founded, but that are not required to be total.

I. Linear $(2, 2)$ -categories, braidings and duals

II. Decreasing order operators

III. Termination heuristics in particular linear $(2, 2)$ categories

IV. Illustration on the diagrammatic rewriting system \mathcal{KLR}

I. Linear $(2, 2)$ -categories, braidings and duals

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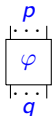
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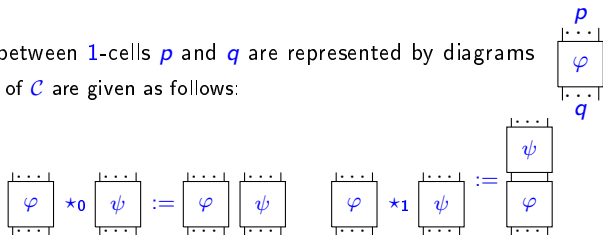
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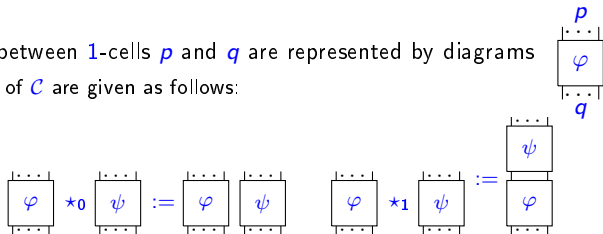
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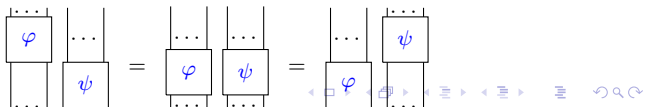
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- ▶ modulo the **exchange law** of \mathcal{C} , diagrammatically depicted as



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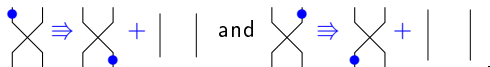
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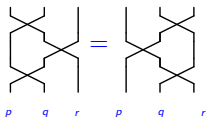
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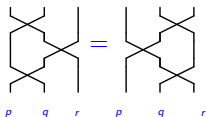
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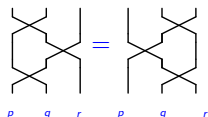
- ▶ Let $p : x \rightarrow y$ be a 1-cell of \mathcal{C} . We say that a 1-cell $q : y \rightarrow x$ is a **left-adjoint** of p , denoted by $q = \hat{p}$ if there exists 2-cells $\varepsilon : p \star_0 \hat{p} \Rightarrow 1_y$ and $\eta : 1_x \Rightarrow \hat{p} \star_0 p$ respectively represented by

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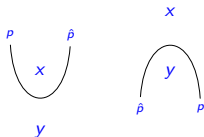
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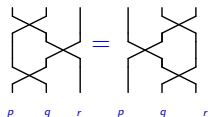


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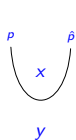
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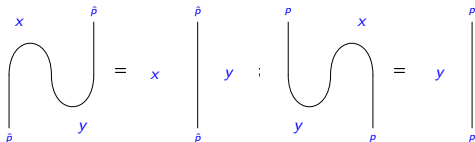


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- Given a pair of 1-cells $p, q : x \rightarrow y$ in \mathcal{C} with chosen biadjoints $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$, then for any 2-cell $\alpha : p \Rightarrow q$, we construct two duals ${}^* \alpha$ and $\alpha^* : \hat{q} \Rightarrow \hat{p}$ as follows:

$${}^* \alpha := \begin{array}{c} x \\ \varepsilon_q \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \hat{p} \\ \eta_p \\ \text{---} \text{---} \text{---} \\ \hat{q} \\ y \end{array}$$

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$${}^*\alpha := \text{Diagram 1} \qquad \alpha^* := \text{Diagram 2}$$

- A 2-cell $\alpha : p \Rightarrow q$ is said **cyclic** if the equation ${}^*\alpha = \alpha^*$ is satisfied, or either of the equivalent conditions ${}^{**}\alpha = \alpha$ or $\alpha^{**} = \alpha$ are satisfied, yielding relations of the form

$$\begin{array}{ccc} \begin{array}{c} p \quad \hat{q} \\ | \quad | \\ \alpha \\ | \quad | \\ \eta_q \quad y \end{array} & = & \begin{array}{c} p \quad \hat{q} \\ | \quad | \\ \alpha^* \\ | \quad | \\ \eta_p \quad y \end{array} \\ \begin{array}{c} \hat{q} \quad p \\ | \quad | \\ \alpha \\ | \quad | \\ \hat{\varepsilon}_q \quad x \end{array} & = & \begin{array}{c} \hat{q} \quad p \\ | \quad | \\ \alpha^* \\ | \quad | \\ \hat{\varepsilon}_p \quad x \end{array} \end{array} \quad (1)$$

Cyclic 2-cells

- Given a pair of 1-cells $p, q : x \rightarrow y$ in \mathcal{C} with chosen biadjoints $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$, then for any 2-cell $\alpha : p \Rightarrow q$, we construct two duals ${}^*\alpha$ and $\alpha^* : \hat{q} \Rightarrow \hat{p}$ as follows:

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$$\text{Diagram 1} = \text{Diagram 2} \qquad \text{Diagram 3} = \text{Diagram 4} \qquad (1)$$

- A linear $(2, 2)$ -category \mathcal{C} in which any 2-cell α is cyclic is called a **pivotal category**.

II. Decreasing order operators

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- ▶ Given a DRS Σ , one defines a **decreasing order operator** (DOO) for Σ as a family of functions $\Phi_{p,q} : \Sigma_2(p,q) \rightarrow \mathbb{N}^{m(p,q)} \times \mathbb{Z}$ indexed by 1-cells p and q , satisfying:

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 - ▶ For any 3-cell $\alpha : D_1 \Rightarrow D_2$ with D_1, D_2 in $\Sigma_2(p, q)$, the function $\Phi_{p,q}$ satisfy

$$\Phi_{p,q}(D_1) > \Phi_{p,q}(D')$$

where $>$ is the lexicographic order on $\mathbb{N}^{m(p,q)} \times \mathbb{Z}$ and D' is a monomial in D_2 . We denote this by $D_1 >_{\text{lex}} D_2$.

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 - ▶ The $\Phi_{p,q}$ are stable by context: for any D_1 and D_2 in $\Sigma_2(p, q)$ and any context C of Σ , if $D_1 >_{\text{lex}} D_2$, then $C[D_1] >_{\text{lex}} C[D_2]$.

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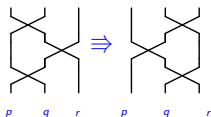
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III. Termination heuristics in particular linear $(2, 2)$ categories

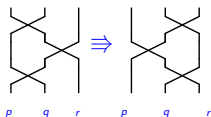
Termination with braid relations

- ▶ Let \mathbf{Crs} be the DRS having: only one 0-cell, a set of generating 1-cells \mathbf{Crs}_1 , for 2-cells the braidings $\sigma_{p,q}$ for each p and q in \mathbf{Crs}_1 , and 3-cells as follows:



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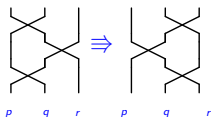
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- ▶ \mathbf{Crs} is terminating by the DOO $\Phi_{p,q}$ counting the number $yb(D)$ of occurrences of 2-cells $\sigma_{p,q} \star_0 id_r$ in a diagram D , for p, q and r in \mathbf{Crs}_1 .

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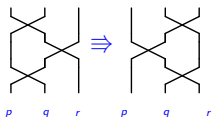
- \mathbf{Crs} is terminating by the DOO $\Phi_{p,q}$ counting the number $yb(D)$ of occurrences of 2-cells $\sigma_{p,q} \star_0 id_r$ in a diagram D , for p, q and r in \mathbf{Crs}_1 .
- Let \mathbf{Crs}' be the DRS defined by

$$\mathbf{Crs}' = \mathbf{Crs} \cup \left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\}$$

The diagram set consists of two diagrams: the first is a crossing of strands p and q , and the second is two parallel vertical strands p and q . They are connected by an equivalence symbol.

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- ▶ Let \mathbf{Crs}' be the DRS defined by

$$\mathbf{Crs}' = \mathbf{Crs} \cup \left\{ \begin{array}{c} \text{Diagram with two crossings} \\ \Rightarrow \\ \text{Two parallel strands} \end{array} \right\}$$

The equation defines \mathbf{Crs}' as the union of \mathbf{Crs} and a set of diagrams. The diagram shown is a braid relation where two crossings are reduced to two parallel strands. The strands are labeled p and q at the bottom.

- ▶ We add as first component to the $\Phi_{p,q}$ defined for \mathbf{Crs} a component counting the number of crossings of the diagrams.

Termination with braid relations and additional 2-cells

- ▶ Let $\mathbf{Crs}^{\text{add}}$ be a DRS defined by

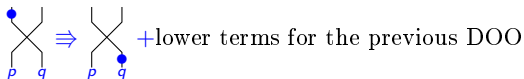
$$\mathbf{Crs}^{\text{add}} = (\mathbf{Crs}'_0, \mathbf{Crs}'_1, \mathbf{Crs}'_2 \cup \left\{ \begin{array}{c} q \\ \bullet \\ \alpha \\ \bullet \\ q \end{array} \text{ for } q \text{ in } \mathbf{Crs}'_1 \right\}, \mathbf{Crs}'_3)$$

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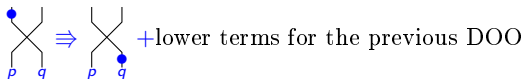


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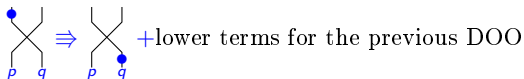
- ▶ We define a new DOO as follows: for $p, q \in \mathbf{Crs}'_1$, we set $m := \max(\ell(p), \ell(q))$. We add to $\Phi_{p, q}$ the components $(c_k(D))_{1 \leq k \leq m}$ defined by:

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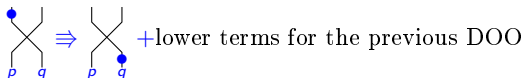
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 - ▶ 0 if there is no \bullet on the k -th strand and if the k -strand is not a **through strand**, but this can not occur with only braidings.

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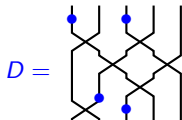
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- ▶ Assume that $\mathbf{Crs}^{\text{add}}$ admits a 3-cell of the following form



\Rightarrow +lower terms for the previous DOO

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 - ▶ 0 if there is no \bullet on the k -th strand and if the k -strand is not a **through strand**, but this can not occur with only braidings.
 - ▶ the number of crossings below the upper dot of the k -th strand.
- ▶ **Example.** For



Example: the nil Hecke algebra

- ▶ For $n \in \mathbb{N}$, let us consider the *Nil-Hecke algebra* \mathcal{NH}_n^0 which is a \mathbb{K} -algebra for a field \mathbb{K} defined by:

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$$\begin{array}{c} \bullet \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \bullet \end{array} + \left| \begin{array}{c} \\ \\ \\ \end{array} \right|$$

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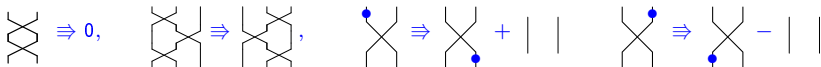
- $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$ form a linear $(2, 2)$ -category with only one 0-cell, the 1-cells are permutations and 2-cells are braiding diagrams.

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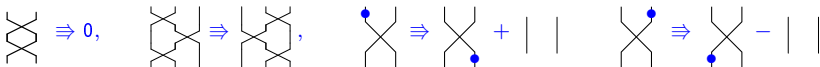
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- $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$ form a linear $(2, 2)$ -category with only one 0-cell, the 1-cells are permutations and 2-cells are braiding diagrams.
- Let Σ be a DRS presenting $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$ with relations oriented as above.
- We prove that Σ is terminating using the following DOO: for a given diagram D in $\mathcal{NH}_n^0(\sigma, \tau)$,

$$\Phi_{\sigma, \tau}(D) = (c(D), \text{yb}(D), c_1(D), \dots, c_n(D)).$$

Termination with adjunctions

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$$\text{cap} \cdot \alpha \Rightarrow | \Leftarrow \text{cup} \cdot \alpha$$

- ▶ The DRS given by these orientations is not confluent: the first Knuth-Bendix step imposes to add the following relations:

$$\text{cap} \cdot \alpha = \text{cup} \cdot \alpha, \quad \text{cup} \cdot \alpha = \text{cap} \cdot \alpha$$

Prototypical example: the 3-polygraph of pearls

► Let **Pearl** be the DRS defined by:

► only one 0-cell $*$;

► only one 1-cell p ;

► generating 2-cells: , , ;

► the following 3-cells:

$$\text{S-shaped curve} \Rightarrow \text{vertical line}, \quad \text{U-shaped curve} \Rightarrow \text{vertical line}, \quad \text{inverted U with dots} \Rightarrow \text{U with dots}, \quad \text{U with dots} \Rightarrow \text{U with dots}.$$

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$$\text{U} \Rightarrow \text{I}, \quad \text{U} \Rightarrow \text{I}, \quad \text{U} \Rightarrow \text{U}, \quad \text{U} \Rightarrow \text{U}.$$

► **Pearl** is terminating, using the following DOO $\Phi_{p,p}(D) = (I(D), \text{l-dot}(D))$ where:

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▶ $I(D)$ corresponds to the number of caps and cups in D ;

▶ $\text{l-dot}(D)$ corresponds to the number of positively left-dotted caps and cups, that is the number of elements  and  (with at least one \bullet) appearing in D with the convention

$$\text{l-dot} \left(\text{cap}^{\bullet} \right) = \text{l-dot} \left(\text{cup}^{\bullet} \right) := n$$

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

- ▶ generating 2-cells: 

- ▶ the following 3-cells:

$$\text{cup} \Rightarrow \text{line}, \quad \text{cap} \Rightarrow \text{line}, \quad \text{cap} \Rightarrow \text{cap}, \quad \text{cup} \Rightarrow \text{cup}.$$

- ▶ **Pearl** is terminating, using the following DOO $\Phi_{p,p}(D) = (I(D), \text{l-dot}(D))$ where:

- ▶ $I(D)$ corresponds to the number of caps and cups in D ;

- ▶ $\text{l-dot}(D)$ corresponds to the number of positively left-dotted caps and cups, that is the number of elements  and  (with at least one \bullet) appearing in D with the convention

$$\text{l-dot} \left(\text{cap}^{\bullet} \right) = \text{l-dot} \left(\text{cup}^{\bullet} \right) := n$$

- ▶ Adding a \star_0 and \star_1 -context to D , we add a constant number of cups and caps, and $\text{l-dot}(D)$ can not increase since a dot cannot move from right of a cap/cup to its left even by adding a context

Termination or quasi-termination ?

- ▶ If we choose different orientation for the dot move relations, we create rewriting cycles

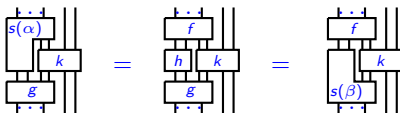


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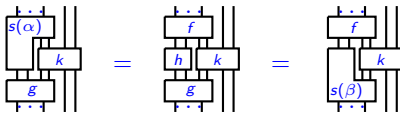
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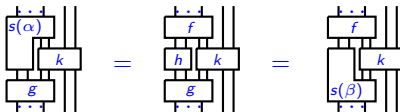
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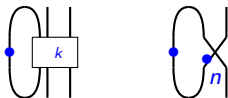


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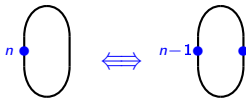
Quasi-termination

- ▶ A DRS Σ is **quasi-terminating** if for each rewriting sequence $(u_n)_{n \in \mathbb{N}}$ of 2-cells of Σ , it contains an infinite number of occurrences of the same 2-cell.
- ▶ Let Σ be a DRS containing the following 3-cells:



Σ is not terminating, one wants to study its quasi-termination.

- ▶ A **quasi-reduced** monomial in Σ is a monomial on which we can only apply the rules



- ▶ We may prove that Σ is quasi-terminating by constructing a DOO on the sets $Q\text{-red}(\Sigma_2(p, q))$ of quasi-reduced monomials between two 1-cells p and q .
 - ▶ This DOO does not take into account the number of left-dotted cups and caps.
 - ▶ It ensures that there is no other obstruction to termination than the bubble cycles.

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 - ▶ (in a \mathbb{Z} -graded context, some relations making the degree decrease with a lower bound on the degree under which all diagrams are zero).
- ▶ **Proposition.** There is a DRS Σ presenting \mathcal{C} in which the relations are oriented in such a way that $\Phi_{p,q}(s(\alpha)) > \Phi_{p,q}(t(\alpha))$ for a DOO of Σ constructed as above and any 3-cell α , and thus Σ is terminating.

IV. Illustration on the linear $(3, 2)$ -polygraph \mathcal{KLR}

The linear $(3, 2)$ -polygraph \mathcal{KLR}

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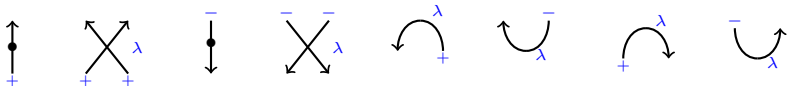
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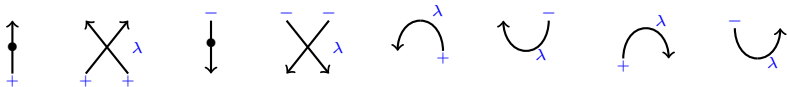
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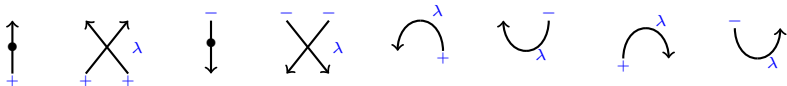
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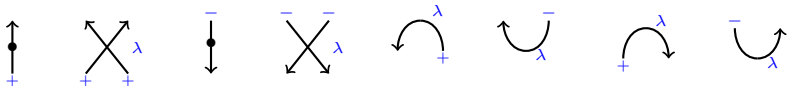
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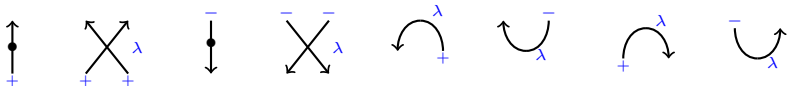


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$$n \begin{array}{c} \circlearrowright \\ \bullet \\ \lambda \end{array} \Rightarrow \begin{cases} 1_{1_\lambda} & \text{if } n = h - 1 \\ 0 & \text{if } n < h - 1 \end{cases}$$

$$\lambda \begin{array}{c} \circlearrowleft \\ \bullet \\ n \end{array} \Rightarrow \begin{cases} 1_{1_\lambda} & \text{if } n = -h - 1 \\ 0 & \text{if } n < -h - 1 \end{cases}$$

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► The 3-cells in \mathcal{KLR}_3 are given by:

► the infinite Grassmannian relation: for any $\lambda \in X$ and $\alpha > 0$,

$$h-1+\alpha \quad \begin{array}{c} \circlearrowleft \\ i \end{array} \lambda \quad \Rightarrow \quad - \sum_{l=1}^{\alpha} h-1+\alpha-l \quad \begin{array}{c} \circlearrowleft \\ i \end{array} \lambda \quad \begin{array}{c} \circlearrowleft \\ i \end{array} \quad -h-1+l$$

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$$\text{Diagram 1} \Rightarrow -\uparrow\downarrow^\lambda + \sum_{n=0}^{h-1} \sum_{r \geq 0} \text{Diagram 2} \quad , \quad \text{Diagram 3} \Rightarrow -\downarrow\uparrow^\lambda + \sum_{n=0}^{-h-1} \sum_{r \geq 0} \text{Diagram 4}$$

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► Some " \mathfrak{sl}_2 " 3-cells:

$$\begin{aligned} \text{Diagram 1} &\Rightarrow \sum_{n=0}^h \text{Diagram 2} \\ \text{Diagram 3} &\Rightarrow - \sum_{n=0}^{-h} \text{Diagram 4} \\ \text{Diagram 5} &\Rightarrow - \sum_{n=0}^{-h} \text{Diagram 6} \\ \text{Diagram 7} &\Rightarrow \sum_{n=0}^h \text{Diagram 8} \end{aligned}$$

The linear $(3, 2)$ -polygraph \mathcal{KLR}

- ▶ \mathcal{KLR} is terminating using the following DOO:

$$\begin{array}{ll} \Phi_{\varepsilon, \varepsilon'} : \mathcal{KLR}_2(\varepsilon, \varepsilon') & \rightarrow \mathbb{N}^{m+4} \times \mathbb{Z} \\ D & \mapsto (c(D), c_1(D), \dots, c_m(D), \text{ybg}(D), I(D), \text{-dot}(D), \text{deg}_b(D)) \end{array}$$

with:

- ▶ $c(D)$ is the number of crossings between strands in D ;
- ▶ for $1 \leq k \leq m$, $c_k(D)$ is defined as above;
- ▶ $\text{ybg}(D)$ defined as above;
- ▶ $I(D)$ corresponds to the number of rightward caps and leftward cups that appear in D ;
- ▶ $\text{-dot}(D)$ corresponds to the number of positively leftward dotted caps and cups as described above.

- ▶
$$\text{deg}_b(D) := \begin{cases} \#\{\text{bubbles in } D\} + \sum_{\pi \text{ clockwise bubble in } D} \text{deg}(\pi) & \text{if } D \text{ is a diagram with} \\ 0 & \text{if } D \text{ is a diagram without} \\ -\infty & \text{if } D = 0. \end{cases}$$

Conclusion

- ▶ We presented heuristics to prove termination of some DRS presenting diagrammatic algebras coming from representation theory.
- ▶ The next question to study is confluence of these DRS.
 - ▶ The diagrammatic structure yield a combinatorial explosion for computation of critical pairs, as for instance isotopy relations.
 - ▶ Isotopy should not be considered as rewrite rules, but as equations we have to take into account when rewriting.
 - ▶ Develop a context of **rewriting modulo isotopy**, and obtain linear bases and coherence results in that setting.