A convergent presentation for the simply-laced KLR algebras and the PBW property

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Motivation : categorification of quantum groups

- In higher representation theory, a natural way to study an algebraic structure is to build a categorification of it.

- These categorifications are higher dimensional categories whose split Grothendieck group is isomorphic to the aforegiven structure.

- This work is about categorification of quantum groups associated with symmetrizable Kac-Moody algebras,
    - following the work of Khovanov and Lauda,'08 or Rouquier,'08.

Objective : find bases of the KLR algebras

- Khovanov and Lauda,'08 built categorifications of these quantum groups which are $\mathbf{2}$-categories.
- The family of *KLR algebras* appear naturally.
  - They act on certain endomorphism spaces of $\mathbf{2}$-cells in these categories.
- To establish that their $\mathbf{2}$-categories are real categorifications, they used a property of non-degeneracy of a diagrammatic calculus.
- This non-degeneracy is proved by finding explicit bases of the spaces of $\mathbf{2}$-cells in the $\mathbf{2}$-categories.
- Explicit bases of the KLR algebras are used to describe bases of some of these spaces of $\mathbf{2}$-cells.

Results presented

- We will construct *linear $(3,2)$-polygraphs* that present the simply-laced KLR algebras.

- We will prove the following result :

Theorem

The linear $(3,2)$-polygraphs KLR are terminating and confluent.

- Application : we obtain a rewriting proof of the following algebraic result obtained by Khovanov and Lauda :

Corollary

The simply-laced KLR algebras admit Poincaré-Birkhoff-Witt bases

- Let Γ be a graph whose set of vertices is denoted by I, and $\mathbb{K}$ any field.
- A Cartan datum $(I, \cdot)$ consists of a finite set I and a bilinear form on $\mathbb{Z}[I]$ the free group generated by I, taking values in $\mathbb{Z}$ such that :
  - $i \cdot i \in \{2, 4, 6, \ldots\}$ for any $i \in I$,
  - $-d_{i,j} := 2\frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \ldots\}$ for any $i \neq j \in I$.
- Such a Cartan datum is said simply-laced if the two following conditions hold :
  - For any $i \in I$, $i \cdot i = 2$,
  - For any $i, j \in I$, $i \cdot j \in \{0, -1\}$.

The set Seq($\mathcal{V}$)

- We set $\mathcal{V} = \sum_{i \in I} \nu_i . i \in \mathbb{N}[I]$ an element of the free semi-group generated by I.
- We put $m := |\mathcal{V}| = \sum \mathcal{V}_i$.
- Let's also consider the set $\mathrm{Seq}(\mathcal{V})$ which consists of all sequences of vertices of Γ with length m in which the vertex i appears exactly $\mathcal{V}_i$ times.
    - For instance, $\mathrm{Seq}(2i + j) = \{iij, iji, jii\}$.

- Let $Q = (Q_{i,j})_{i,j \in I}$ be a matrix with coefficients in $\mathbb{K}[u,v]$ with $Q_{i.i} = 0$ for all $i \in I$.
- Define the $\mathbb{K}$-algebra $H_{\mathcal{V}}(Q)$ by
  - generators : $1_{\mathsf{i}}, x_{k,\mathsf{i}}$ for $k \in \{1,\ldots,n\}$ and $\tau_{k,\mathsf{i}}$ for $k \in \{1,\ldots,n-1\}$ and $\mathsf{i} \in \mathrm{Seq}(\mathcal{V})$.
  - relations :

(1) $1_{\mathsf{i}}1_{\mathsf{j}} = \delta_{\mathsf{i},\mathsf{j}}1_{\mathsf{i}}$

(2) $\tau_{k,\mathsf{i}} = 1_{s_k(\mathsf{i})}\tau_{k,\mathsf{i}}1_{\mathsf{i}}$

(3) $x_{k,\mathsf{i}} = 1_{\mathsf{i}}x_{k,\mathsf{i}}1_{\mathsf{i}}$

(4) $x_{k,\mathsf{i}}x_{l,\mathsf{i}} = x_{l,\mathsf{i}}x_{k,\mathsf{i}}$

(5) $\tau_{k,s_k(\mathsf{i})}\tau_{k,\mathsf{i}} = Q_{i_k,i_{k+1}}(x_{k,\mathsf{i}}, x_{k+1,\mathsf{i}})$

(6) $\tau_{k,s_l(\mathsf{i})}\tau_{l,\mathsf{i}} = \tau_{l,s_k(\mathsf{i})}\tau_{k,\mathsf{i}}$ if $|k-l|>1$

(7) $\tau_{k,\mathsf{i}}x_{l,\mathsf{i}} - x_{s_k(l),s_k(\mathsf{i})}\tau_{k,\mathsf{i}} = \begin{cases} -1_{\mathsf{i}} & \text{if} \quad l=k \text{ and } i_k = i_{k+1} \\ 1_{\mathsf{i}} & \text{if} \quad l = k+1 \text{ and } i_k = i_{k+1} \\ 0 & \text{otherwise} \end{cases}$

(8) $\tau_{k+1,s_k s_{k+1}(\mathsf{i})}\tau_{k,s_{k+1}(\mathsf{i})}\tau_{k+1,\mathsf{i}} - \tau_{k,s_{k+1}s_k(\mathsf{i})}\tau_{k+1,s_k(\mathsf{i})}\tau_{k,\mathsf{i}} = \begin{cases} \frac{Q_{i_k,i_{k+1}}(x_{k+2,\mathsf{i}},x_{k+1,\mathsf{i}}) - Q_{i_k,i_{k+1}}(x_{k,\mathsf{i}},x_{k+1,\mathsf{i}})}{x_{k+2,\mathsf{i}}-x_{k,\mathsf{i}}} & \text{if} \quad i_k = i_{k+2} \\ 0 & \text{otherwise} \end{cases}$

- We will consider the definition of Khovanov and Lauda that gives the following specialization :

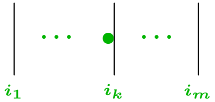
$$Q_{i,j}(u,v) = u^{d_{i,j}} + v^{d_{j,i}} \quad \forall \quad i,j \in I$$

- We will consider the case of a *simply-laced graph* : that is a graph with no loops nor multiple edges.
  - From such a graph, we define a *simply-laced Cartan datum* as follows : let $\cdot$ be a bilinear form on $\mathbb{Z}[I]$ such that :

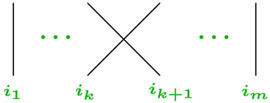
$$\begin{cases} i \cdot i = 2 & \\ i \cdot j = -1 & \text{if there is an edge in } \Gamma \text{ from } i \text{ to } j \\ i \cdot j = 0 & \text{otherwise} \end{cases}$$

  - In this case, we have the coefficients $d_{i,j}$ and $d_{j,i}$ all equal to 1 when $i \cdot j = -1$.

- Khovanov and Lauda provided a diagrammatic approach for these algebras : for $\mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$, we represent the generators by the diagrams :

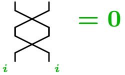
  - $x_{k,\mathbf{i}}$ = 

for $1 \leq k \leq m, \mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$

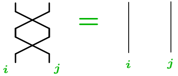
  - $\tau_{k,\mathbf{i}}$ = 

for $1 \leq k \leq m-1, \mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$

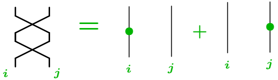
- The local relations are represented by :
  i) For any $i \in I$,



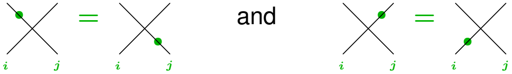
  ii) For any $i, j \in I$ such that $i \cdot j = 0$,



  iii) For any $i, j \in I$ such that $i \cdot j = -1$,

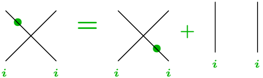


  iv) For any $i, j \in I$,

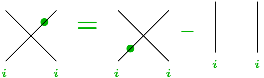


- The local relations are represented by :

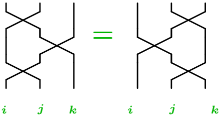
  v) For any $i \in I$,



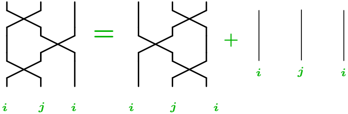
and



  vi) For any $i, j, k \in I$, and unless $i = k$ and $i \cdot j = -1$,



  vii) For any $i, j \in I$ such that $i \cdot j = -1$,



- They correspond respectively to the relations (5), (7) and (8).

- We denote by $R(\mathcal{V})$ the aforegiven algebra : we call it *the simply-laced KLR Algebra*.

- For **i** and **j** $\in \mathrm{Seq}(\mathcal{V})$, we define the set ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$ as the set of "braid-like diagrams" from **i** to **j**, that is :
    - Each strand is labelled by a vertex of Γ ;
    - A brand does not intersect with itself ;
    - One has to read **i** (resp. **j**) at the bottom (resp. at the top) of the diagram

- These algebras can be seen as 2-categories with :
    - One 0-cell,
    - The 1-cells are the elements of $\mathrm{Seq}(\mathcal{V})$,
    - The 2-cells between two sequences **i** and **i** are ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$.

The structure of linear $(2,2)$-categories

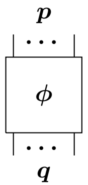
- The space of 2-cells ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$ is a vector space.

- The simply-laced KLR algebras are *linear $(2,2)$-categories*.

- Linear $(2,2)$-categories are categories enriched in linear categories.
  - Explicitely, such a category $\mathcal{C}$ has 0-cells $\mathcal{C}_0$, 1-cells $\mathcal{C}_1$ and 2-cells $\mathcal{C}_2$
  - For every p and q in $\mathcal{C}_1$, the space of 2-cells $\mathcal{C}_2(p,q)$ between p and q is a vector space.

- According to Alleaume '16, these linear $(2,2)$-categories can be presented by rewriting systems called linear $(3,2)$-polygraphs.

- In those rewriting systems, the generating 2-cells have the form of a circuit as follows :

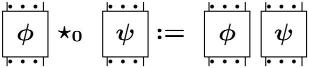


where p and q are two 1-cells of the category.

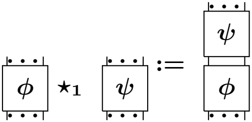
Compositions of 2-cells in linear $(3, 2)$-polygraphs

- These generators can be composed in two ways

Horizontally



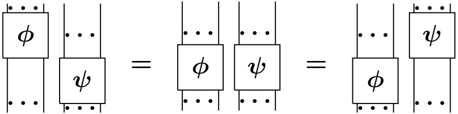
Vertically



- All these compositions are made modulo the exchange law of the $\mathbf{2}$-category, that is for every $\mathbf{2}$-cells ϕ_1, ϕ_2, ψ_1, ψ_2 one has

$$(\psi_1 \star_0 \phi_1) \star_1 (\psi_2 \star_0 \phi_2) = (\psi_1 \star_1 \psi_2) \star_0 (\phi_1 \star_1 \phi_2)$$

which is diagrammatically depicted as :



The linear structure

- One can also make linear combinations of these circuits with scalars in a ground field $\mathbb{K}$. An element of the form

$\boldsymbol{\lambda}$ 

  where ϕ is a $\mathbf{2}$-cell obtained with the previous compositions of generating $\mathbf{2}$-cells and $\boldsymbol{\lambda} \in \mathbb{K}$ is called a *monomial* in the linear $(\mathbf{3}, \mathbf{2})$-polygraph.

- Given a $\mathbf{2}$-cell ϕ, it can be uniquely decomposed into a sum of monomials $\phi = \sum \phi_i$, called the *monomial decomposition* of ϕ.

- The *support* of ϕ is the set of all the ϕ_i in that decomposition.

- A *rewriting step* of Σ is a 3-cell of the form

$$\lambda m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4 + u \Rrightarrow$$

$$\lambda m_1 \star_1 (m_2 \star_0 t_2(\alpha) \star_0 m_3) \star_1 m_4 + u$$

where $s_2(\alpha)$ and $t_2(\alpha)$ are two parallel 2-cells such that the monomial $\lambda m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4$ does not appear in the monomial decomposition of u.

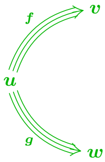
- A *rewriting sequence* of Σ is a finite or infinite sequence :

$$u_0 \Rrightarrow \cdots \Rrightarrow u_n \Rrightarrow \cdots$$

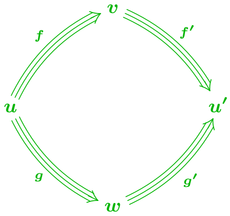
of rewriting steps of Σ.

- A *normal form* is a 2-cell that can't be reduced by any rewriting step.

- A *branching* of Σ is



- A branching is *confluent* if it can be completed by rewriting sequences f' and g' as follows :



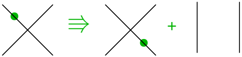
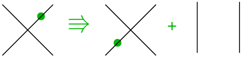
- A *local branching* of Σ is a pair of rewriting steps of Σ with the same 2-source.

Termination, confluence, left monomiality

- A linear $(3, 2)$-polygraph is :
  - *confluent* (*resp. locally confluent*) if all its (resp. local) branchings are confluent.
  - *terminating* if it has no infinite rewriting sequence.
  - *left monomial* is every source of a 3-cell in Σ is a monomial.

- **Example.** Here, an example of linear $(3, 2)$-polygraph with one 0-cell, one 1-cell, two generating 2-cells

and two 3-cells :

 and 

- In this setting, we have a version of classic rewriting results such as Noetherian's induction principle and Newman's lemma.

Proposition

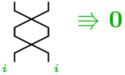
A terminating linear $(3, 2)$-polygraph is confluent if and only if all its critical branchings are confluent.

Proposition (Alleaume,'16)

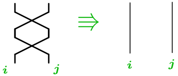
Let Σ be a confluent and terminating left-monomial linear $(3, 2)$-polygraph and $\mathcal{C}$ be the linear $(2, 2)$-category presented by Σ. Then, for any 1-cells u and v of $\mathcal{C}$ with same 0-source and 0-target, the set of monomials of Σ in normal form from u to v gives a basis of $\mathcal{C}(u, v)$.

- We define the linear $(3,2)$-polygraphs KLR by :
  - One 0-cell $\{*\}$
  - The 1-cells are $\mathbf{i} \in \mathrm{Seq}(\mathcal{V})$ so that the generating 1-cells are $i \in I$
  - The 2-cells between two 1-cells $\mathbf{i}$ and $\mathbf{j}$ are given by the braid-like diagrams which link $\mathbf{i}$ to $\mathbf{j}$.
  - The 3-cells are given by the diagrammatic relations oriented as follows.

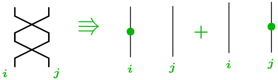
i) For any $i \in I$,



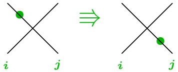
ii) For any $i, j \in I$ such that $i \cdot j = 0$,



iii) For any $i, j \in I$ such that $i \cdot j = -1$,



iv) For any $i, j \in I$,



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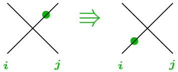
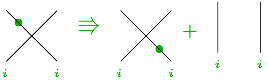


The 3-cells in Γ

v) For any $i \in I$,



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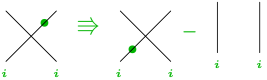
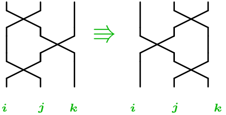
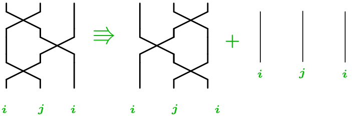


vi) For any $i, j, k \in I$, and unless $i = k$ and $i \cdot j = -1$,



vii) For any $i, j \in I$ such that $i \cdot j = -1$,



- We split the proof in two parts :
  - First of all, we prove that KLR is terminating.
  - Then, we show that it is confluent by examining all the critical branchings.

The idea of the process of termination (Guiraud,'06)

- Each $\mathbf{2}$-cell is seen as an electronical circuit whose components are given by the generating $\mathbf{2}$-cells
- Fix a value for each component ;
  - With this value, each output of the circuit receives a certain intensity of courant.
- The heat produced by a fixed component is calculated this way :
  - A component is arbitrarily chosen.
  - Currents are propagated through the other components to the chosen one.
  - One compute the intensities of currents transmitted when the incoming current is known.

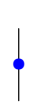
The idea of the process of termination (Guiraud,'06)

- One repeats the same procedure for each component.
- One gets the heat produced by a circuit by summing the heat produced by all its components.
- Two circuits with the same number of inputs and the same number of outputs are compared this way.
- We build a reduction order by comparing all the sources and targets of $\mathbf{2}$-cells following this method.

Generalization of this idea

- Guiraud-Malbos '09 generalized this idea in a categorical framework.
  - The theorem lays on a construction of a derivation d and a 2-functor.
  - They are defined on the generating 2-cells of the polygraph.
  - One has to check that : $X(s\alpha) \geq X(t\alpha)$ and $d(s\alpha) > d(t\alpha)$ for every 3-cell α.
- We adapt this theorem in a linear setting :
  - The conditions we have to check are $X(s\alpha) \geq X(g)$ and $d(s\alpha) > d(g)$ for every $g \in \text{Supp}(t\alpha)$.

- In our case, we define a 2-functor $X : \mathrm{KLR}_2^* \to$ **Ord** on generating 2-cells by :

$X\Big($ $\Big)(i) = i;$ $X\Big($  $\Big)(i) = i{+}1;$ $X\Big($  $\Big)(i,j) = (j{+}1, i) \quad \forall i,j \in \mathbb{N}.$

- We have the following inequalities :

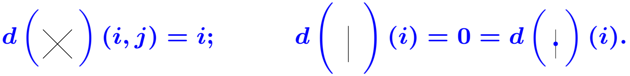
$X\Big($  $\Big)(i,j) = (i{+}1, j{+}1) \geq (i{+}1, j{+}1) = \max\Big(X\Big($  $\Big), X\Big($  $\Big)\Big)(i,j);$

$X\Big($  $\Big)(i,j) = (j+2, i) \geq (j+2, i) = \max\Big(X\Big($  $\Big), X\Big($  $\Big)\Big)(i,j);$

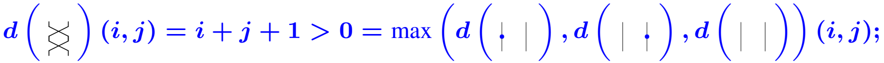
$X\Big($  $\Big) = (j{+}1, i{+}1) \geq (j{+}1, i{+}1) = \max\Big(X\Big($  $\Big)(i,j), X\Big($  $\Big)\Big)(i,j);$

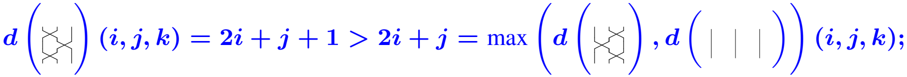
$X\Big($  $\Big)(i,j,k) = (k+2, j+1, i) \geq \max\Big(X\Big($  $\Big), X\Big($  $\Big)\Big)(i,j,k).$

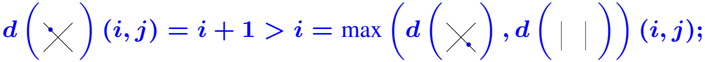
- We now define the derivation d of KLR_2^* into $M_{X,*,\mathbb{Z}}$ given on the generators by

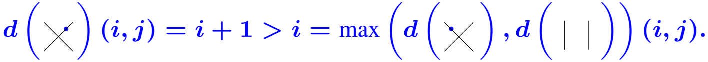


- We can then check the following inequalities :





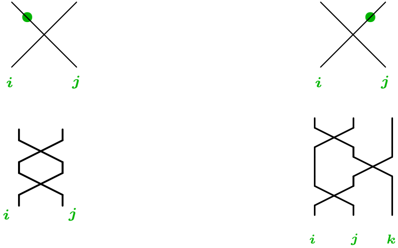




- Thus, KLR is terminating.

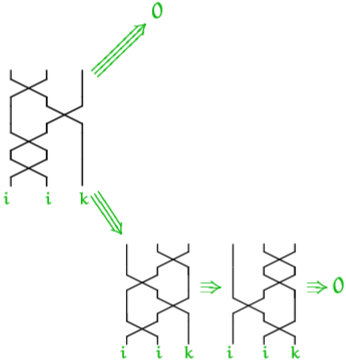
Sources of the $\mathbf{3}$-cells

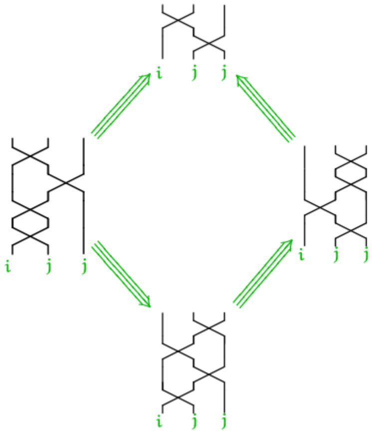
- We have 4 different forms for the sources of $\mathbf{3}$-cells :



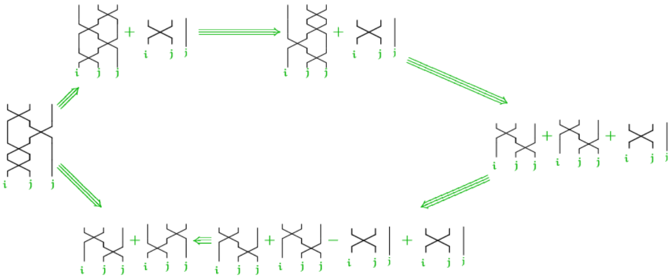
for every $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ in $\boldsymbol{I}$.

  - They depend on the vertices $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ at the bottom.
  - The critical branchings have to be computed for each sequence of vertices and each values of the bilinear form.

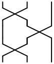
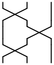
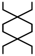
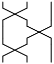
Sequence	iik
Value of $\cdot$	$\mathbf{0}$ or $\mathbf{-1}$
Branching	

Sequence	ijj
Value of $\cdot$	$\mathbf{0}$
Branching	

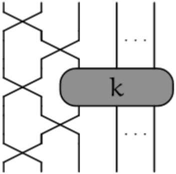
Examples of critical branchings

Sequence	ijj
Value of $\bullet$	-1
Branching	

Families of critical branchings

- There exists 6 main families of critical branchings ;
  - They are characterized by the pair of $\mathbf{2}$-cells which form the branching.
- They are the following ones
  - **Crossings with two dots :**  and 
  - **Triple crossings :**  with itself
  - **Double crossings with dots :**  or  with 
  - **Double Yang-Baxter :**  with itself
  - **Yang-Baxter with crossings :**  with 
  - **Yang-Baxter with dots :**  with  or 

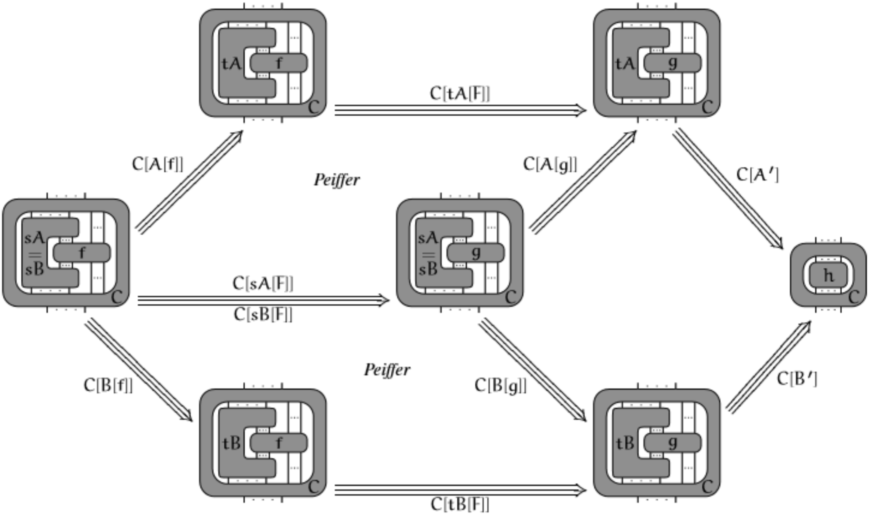
- There is another kind of critical branchings, namely the *right indexations*, that is critical branchings with the form



where k is a diagram that can be plugged in the Yang-Baxter-equation.

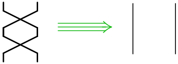
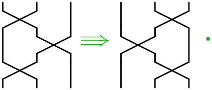
- It was proved by Guiraud and Malbos that it is sufficient to check for the instances k in normal form, according to the following diagram :

The indexed critical branchings

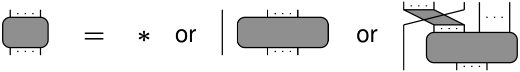


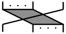
- Thus, we have now to determine which are the normal forms that we can plug in the previous diagram.

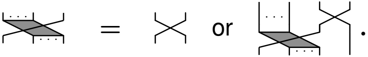
- Guiraud-Malbos '09 made a full study of the normal forms of the 3-polygraph of permutations $\mathbf{\Delta}$ which has
  - One $\mathbf{0}$-cell ;
  - One $\mathbf{1}$-cell ;
  - One $\mathbf{2}$-cell  ;
  - The following two 3-cells :

 and 

- The set of normal forms of that polygraph is given by the set N of 2-cells of Δ^* given by the following inductive graphical scheme :



where  is itself defined inductively by



- The *Coxeter presentation* of the symmetric group $\mathcal{S}_m$ is given by

$$\langle (s_i)_{1 \leq i \leq m-1};\ s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \quad \text{if} |i-j| > 1 \rangle$$

where $s_i = (i \quad i+1) \in \mathcal{S}_m$.

  - *Length* of a permutation = $\min\{r \in \mathbb{N};\ \exists s_{i_1}, \ldots, s_{i_r} \backslash \sigma = s_{i_1} \ldots s_{i_r}\}$

- We can add dots wherever on the diagrams. We consider a map

$$\begin{array}{cccc} f & : & R(\mathcal{V}) & \to & \mathbb{N}^m \\ & & D & \mapsto & (c_1(D), \ldots, c_m(D)) \end{array}$$

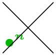
 where for every $1 \leq k \leq m$, $c_k(D)$ is the number of crossing under the upper dot on the k-th strand of D.

- If a diagram D is such that $f(D) > (0, \ldots, 0)$, then it can be reduce by making the dot go down.
  - The result gives a linear combination of diagrams $\sum \lambda_i D_i$ such that for all i, $f(D) > f(D_i)$ for the lexicographic order.

- The monomials in normal form are the normal forms of the polygraph of permutations for which the image by f is 0.

- They correspond to the diagrams :
  - which contain a minimal number of crossings, that is the length of the associated permutation ;
  - with all the elements $\tau_{k+1,s_k s_{k+1}(\mathrm{i})}\tau_{k,s_{k+1}(\mathrm{i})}\tau_{k+1,\mathrm{i}}$ are replaced by $\tau_{k,s_{k+1}s_k(\mathrm{i})}\tau_{k+1,s_k(\mathrm{i})}\tau_{k,\mathrm{i}}$;
  - which contain dots that are all placed at the bottom of the diagram.

- There are two families of normal forms that can be plugged :
  - 

 for all $n \in N$ (just the identity if $n = 0$)
  - 

 for all $n \in \mathbb{N}$

- Let $\{s_{i_1}, \ldots, s_{i_r}\}_{(i_1,\ldots,i_m)\in \mathrm{Seq}(\mathcal{V})}$ be a set of minimal length representative of elements of $\mathcal{S}_n$.
- Rouquier,'08 defined a Poincaré -Birkhoff-Witt property, that is equivalent to the fact that

$$S = \{\tau_{i_1, s_{i_2} \ldots s_{i_r}(\mathbf{j})} \cdots \tau_{i_r, \mathbf{j}} x_{1,\mathbf{j}}^{a_1} \cdots x_{m,\mathbf{j}}^{a_m}\}_{(i_1,\ldots,i_r)\in J, (a_1,\ldots,a_m)\in \mathbb{N}^m, \mathbf{j}\in \mathrm{Seq}(\mathcal{V})}$$

is a basis of the algebra $H_{\mathcal{V}}(Q)$.

- Khovanov and Lauda,'08 looked at a basis for the diagrams with source **i** and target **j**.
  - It contains the diagrams of the required form.

- We proved that the linear $(3,2)$ polygraphs KLR were convergent.
- The set of monomials in normal form of these polygraphs form bases of these algebras.
  - This corresponds exactly to the PBW bases, so we proved the following result :

Corollary

The simply-laced KLR algebras admit Poincaré-Birkhoff-Witt bases

[1]-C. Alleaume, *Rewriting in higher dimensional linear categories and application to the affine Oriented Brauer category*, 2016, arXiv : 1603.02592, Journal of Pure and Applied Algebra (to be published).

[2]- Y. Guiraud, P. Malbos, *Higher dimensional categories with finite derivation type*, 2009, Theory and Applications of Categories Vol 22, p.420–478.

[3]-R. Rouquier, *2-Kac-Moody algebras*, 2008, arXiv :0812.5023.

[4]-M. Khovanov, A. Lauda *A diagrammatic approach to categorification of quantum groups III*, 2018, arXiv :0807.3250.

[5]- Y. Guiraud, *Termination orders for three dimensional rewriting*, 2006, J. Pure Appl. Algebra, 207(2) :341-371.

ANY QUESTIONS ?

Thanks for your attention.