Higher-dimensional linear rewriting

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Linear rewriting : motivations

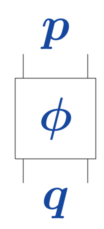
- The theory of linear rewriting has been developed to provide algorithmic methods :
  - to compute linear bases of linear algebraic structures ;
  - to compute homological invariants ;
  - to obtain coherence theorems.
- **Objectives :**
  - to extend the set theoretical 2-dimensional linear rewriting techniques (string rewriting) to higher-dimensional linear rewriting issues.
  - to generalize the notions of (non) commutative Gröbner basis from Bergman, Bokut, Buchberger in higher-dimensional algebraic structures.

Termination and confluence properties in a linear setting

- **Termination :** The definition of the rewriting steps has to take into account the linear setting, otherwise we lose termination.
  Example. With the rule $xy \Rightarrow zx$, then $-xy \Rightarrow -zx$ so
  $$zx = (xy + zx) - xy \Rightarrow (xy + zx) - zx = xy.$$
- **Confluence :** Without termination the critical pair lemma may fail.
  Example. With the rules $x \overset{\alpha}{\Rightarrow} y$ and $y \overset{\beta}{\Rightarrow} -x$, there is no critical branching but a non confluent additive branching :
  $$2y \overset{\alpha+y}{\Leftarrow} x + y \overset{x+\beta}{\Rightarrow} 0$$

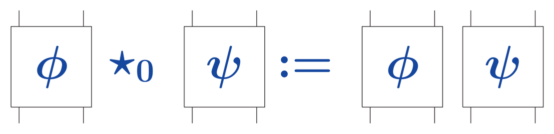
Linear $(2,2)$-categories

- A linear $(2,2)$-category $\mathcal{C}$ over a field $\mathbb{K}$ is a 2-category with :
  - a set $\mathcal{C}_0$ of 0-cells, and a set $\mathcal{C}_1$ of 1-cells denoted by $p, q, \ldots$;
  - a set $\mathcal{C}_2$ of 2-cells such that for every p and q in $\mathcal{C}_1$, the space $\mathcal{C}_2(p,q)$ of 2-cells of source p and target q is a $\mathbb{K}$-vector space ;
    - The map $\star_1 : \mathcal{C}_2(p,q) \times \mathcal{C}_2(q,r) \to \mathcal{C}_2(p,r)$ is bilinear ;
    - Source and target maps are compatible with the linear structure.
- A 2-cell ϕ in $\mathcal{C}$ can be pictured as a circuit as follows :

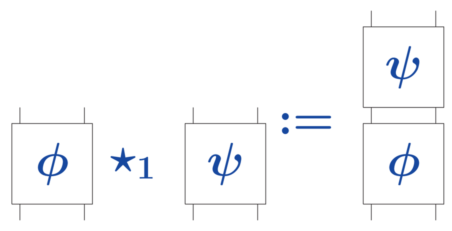


- 2-cells can be composed in two ways :

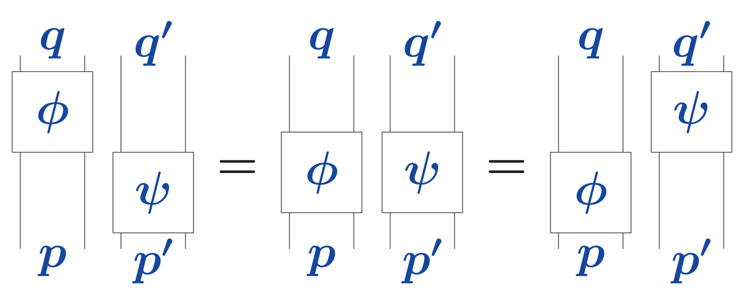
Horizontally



Vertically



- Compositions satisfy the exchange law, diagrammatically depicted as :

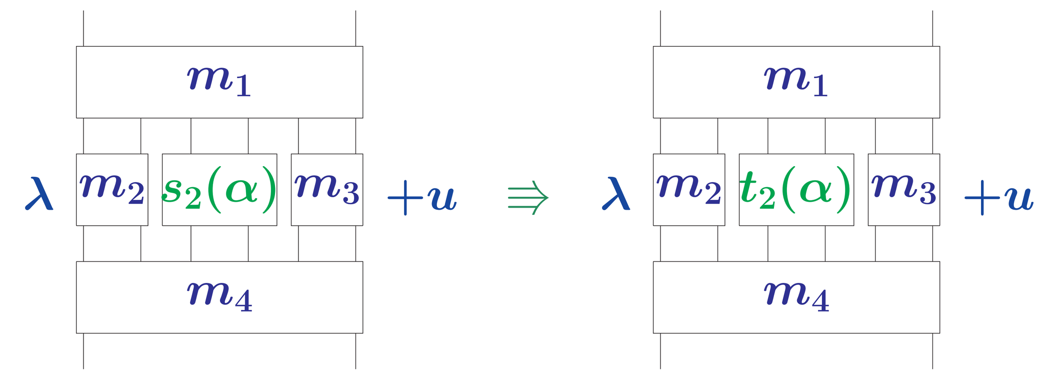


that is, for every 1-cells p, q, p', q' and 2-cells ϕ, ψ, the following relation hold :
$$(\phi \star_0 \mathrm{id}_q) \star_1 (\psi \star_0 \mathrm{id}_{p'}) = (\phi \star_0 \mathrm{id}_{q'}) \star_1 (\psi \star_0 \mathrm{id}_p)$$

- We consider presentations of linear $(2,2)$ -categories given by :
  - generating 1-cells w.r.t $\star_0$;
  - generating 2-cells w.r.t $\star_0$ and $\star_1$.
- A 2-cell ϕ obtained with the previous compositions of generating 2-cells is called a *monomial* in $\mathcal{C}$.
- Given a 2-cell ϕ, it can be uniquely decomposed into a sum of monomials $\phi = \sum \phi_i$, called the *monomial decomposition* of ϕ.
- The *support* of ϕ is the set of all the ϕ_i in that decomposition.

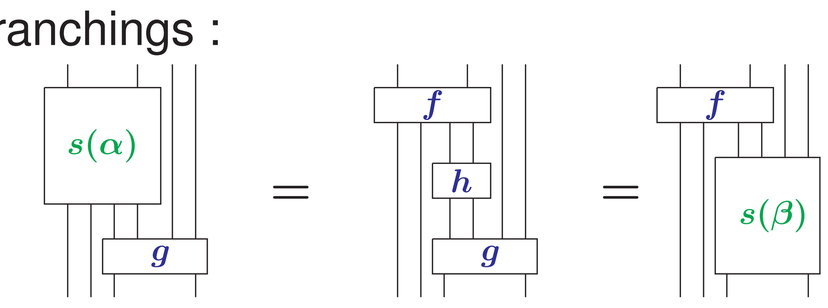
3-dimensional linear rewriting

- A (monomial) rewriting rule is a 3-cell $s(\alpha) \overset{\alpha}{\Rightarrow} t(\alpha)$ where $s(\alpha)$ is a monomial in $\mathcal{C}$.
- We define a rewriting step between two parallel 2-cells as a 3-cell with the following shape :

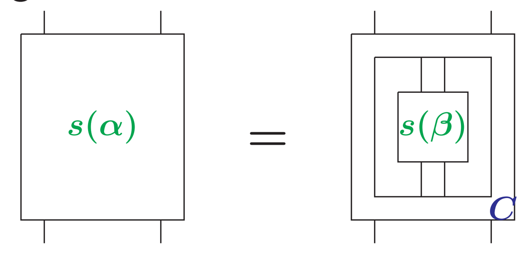


where α is a rewriting rule, and such that the monomial $m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4$ does not appear in the monomial decomposition of u.

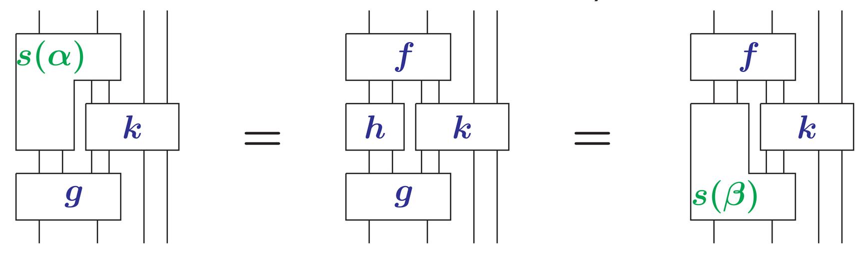
- The set of these rewriting steps defines an abstract rewriting system, called a 3-dimensional linear rewriting system (3-LRS).
- The objective is to study properties such as termination, normal forms, branchings, critical branchings, confluence of this system.
- 3-LRSs present linear $(2,2)$-categories as 2-LRSs present algebras.
- The critical branchings have the following shape :
  - Regular critical branchings :



  - Inclusion critical branchings :



  - Right-indexed (also left-indexed, multi-indexed) critical branchings :

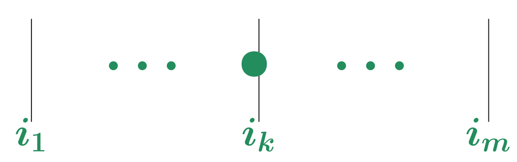


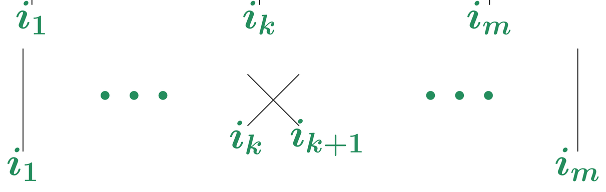
Theorem. *A terminating 3-LRS is confluent if and only if all its critical branchings are confluent.*

Proposition. (Alleaume, '16) *Let Σ be a confluent and terminating 3-LRS presenting a linear $(2,2)$-category $\mathcal{C}$. Then, the set of monomials of $\mathcal{C}_2(p,q)$ in normal form wrt Σ gives a linear basis of $\mathcal{C}_2(p,q)$.*

Application to a family of algebras arising in representation theory

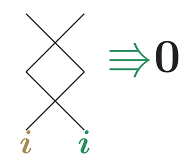
- Let I be a set of vertices of a graph and $\mathcal{V} = \sum_{i \in I} \mathcal{V}_i \cdot i \in \mathbb{N}[I]$. Denote by $\mathrm{Seq}(\mathcal{V})$ the set of sequences of elements of I in which i appears exactly $\mathcal{V}_i$ times and $m := \sum_{i \in I} \mathcal{V}_i$. Fix $\cdot$ a bilinear pairing on I such that $i \cdot j \in \{0, -1\}$ for any i, j. The simply-laced KLR algebra is the $\mathbb{K}$-algebra presented by :
- **generators** :

$x_{k,\mathbf{i}}$ =  for $1 \leq k \leq m, \mathbf{i} = i_1 \ldots i_m \in \mathrm{Seq}(\mathcal{V})$

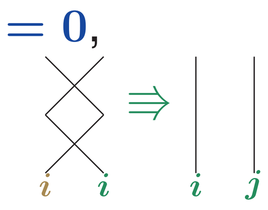
$\tau_{k,\mathbf{i}}$ =  for $1 \leq k \leq m-1, \mathbf{i} = i_1 \ldots i_m \in \mathrm{Seq}(\mathcal{V})$

- **relations** :

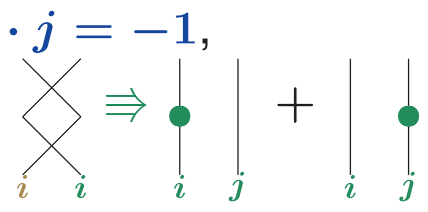
i) For any $i \in I$,



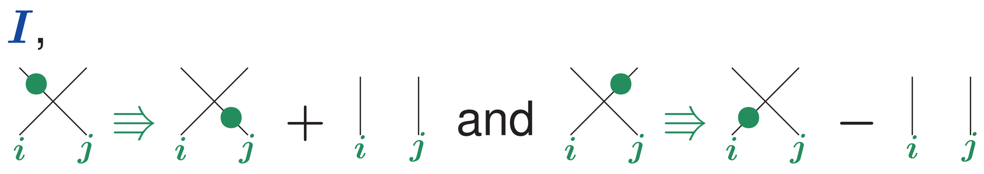
ii) For any $i, j \in I$ such that $i \cdot j = 0$,



iii) For any $i, j \in I$ such that $i \cdot j = -1$,

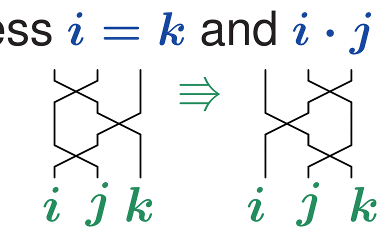


iv) For any $i, j \in I$,

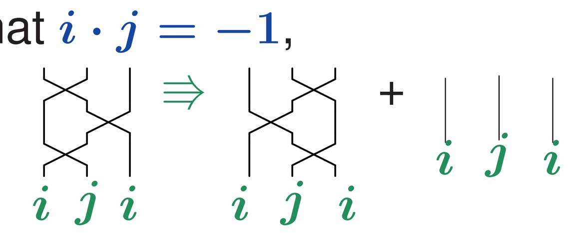


v) For any $i \in I$,

vi) For any $i, j, k \in I$, and unless $i = k$ and $i \cdot j = -1$,



vii) For any $i, j \in I$ such that $i \cdot j = -1$,



- Above oriented relations form a family of 3-LRS $\mathrm{KLR}_{\mathcal{V}}$ presenting the simply-laced KLR algebras.

Theorem. *The 3-LRSs $\mathrm{KLR}_{\mathcal{V}}$ are terminating and confluent. Moreover, diagrams with source **i** and source **j** having a minimal number of crossings and all dots placed at the bottom form a Poincaré-Birkhoff-Witt basis of $\mathrm{KLR}_{\mathcal{V}}(\mathbf{i}, \mathbf{j})$.*

Work in progress

- We would like to extend these methods to higher-dimensional linear categories with an additionnal structure, for categories with duals or pivotal categories in which two isotopic diagrams represent the same 2-cell.
- The isotopy relations may be source of problems for the termination and confluence of our system. We develop a theory of rewriting modulo this equational theory to obtain termination proofs and confluence criteria in that context.

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