Coherence modulo and double groupoids

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joint work with Philippe Malbos

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Plan

I. Introduction and motivations

II. Double groupoids

III. Polygraphs modulo

IV. Coherence modulo

I. Introduction and motivations

Motivations: algebraic context

- Algebraic rewriting: constructive methods to study algebraic structures presented by generators and relations.

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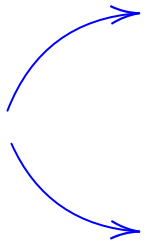

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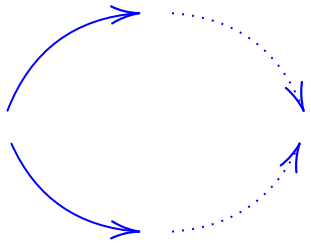

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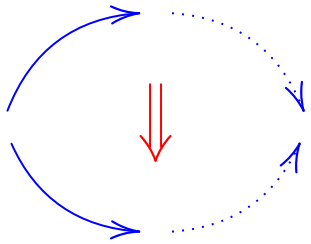

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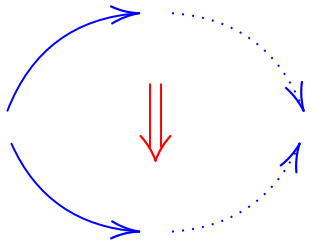

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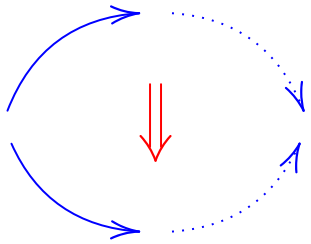

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$$\begin{tikzcd}
 & x \arrow[dd, bend left, "\|"] \\
xx^{-1}x \arrow[ur, bend left, "\alpha_x x"] \arrow[dr, bend right, "x\overline{\alpha_x}"'] & \\
 & x
\end{tikzcd}$$

is an artefact induced by the algebraic structure and should not be considered as a syzygy.

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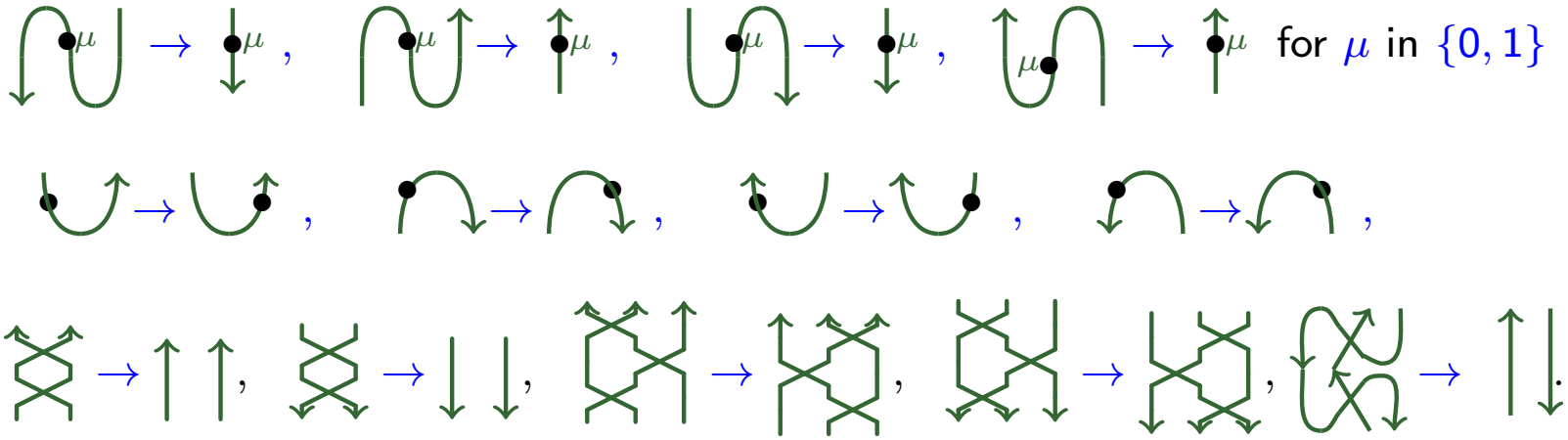

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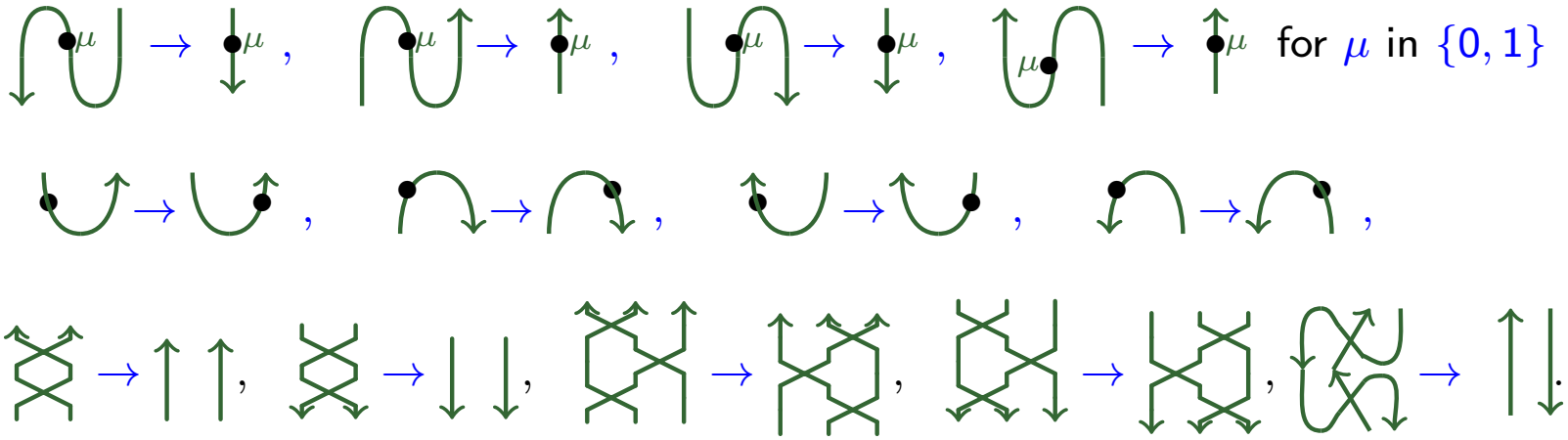

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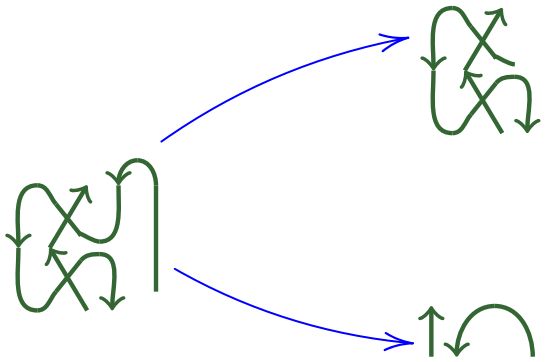

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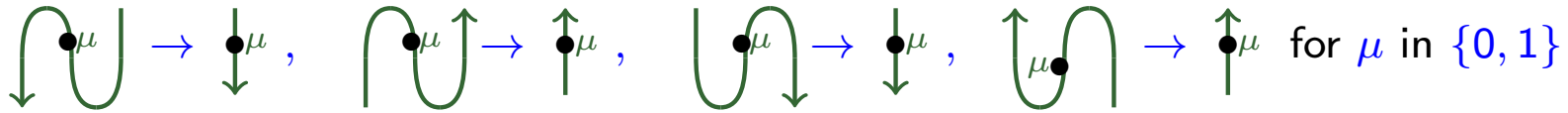
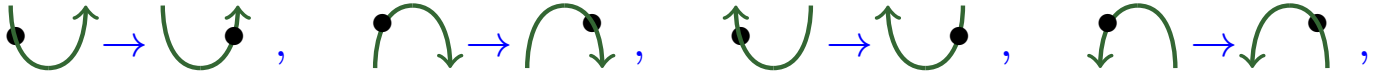
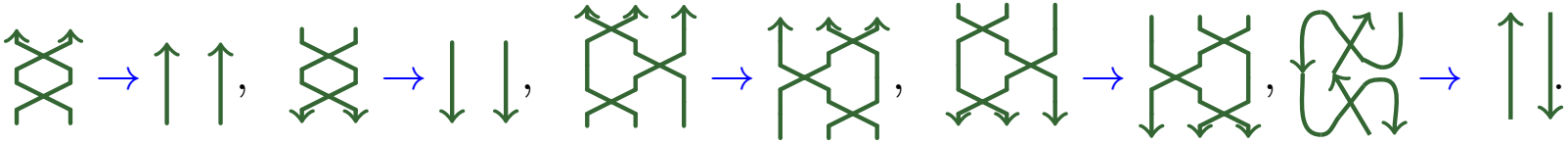


for μ in $\{0, 1\}$





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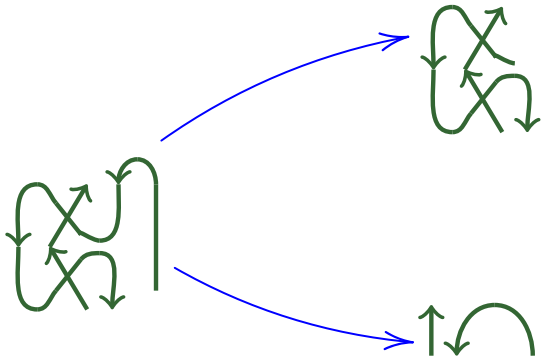

Not confluent !

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- Main interest and results for ${}_E R$.

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- It gives four related categories

$$\mathbf{C}^{vo} := (\mathbf{C}^v, \mathbf{C}^o, \partial^v_{-,0}, \partial^v_{+,0}, \circ^v, i^v_0), \qquad \mathbf{C}^{ho} := (\mathbf{C}^h, \mathbf{C}^o, \partial^h_{-,0}, \partial^h_{+,0}, \circ^h, i^h_0),$$

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  where $\mathbf{C}^{sh}$ is the category $\mathbf{C}_1$ and $\mathbf{C}^{vo}$ is the category $\mathbf{C}_0$.

- Elements of $\mathbf{C}^o$: point cells, elements of $\mathbf{C}^h$ and $\mathbf{C}^v$: horizontal cells and vertical cells.

$$x_1 \xrightarrow{f} x_2 \qquad\qquad \begin{tikzcd} x_1 \arrow[d, "e"'] \\ x_2 \end{tikzcd}$$

Double groupoids

- Elements of $\mathbf{C}_s$ are square cells:

$$\begin{tikzcd}
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, with identities

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- Compositions

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for all x_i,y_i,z_i in $\mathbf{C}^o$, f_i in $\mathbf{C}^h$, e_i,e'_i in $\mathbf{C}^v$ and A, A',B in $\mathbf{C}^s$.

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$$\begin{tikzcd} x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\ y_1 \arrow[r, "g_1"'] & y_2 \arrow[r, "g_2"'] & y_3 \end{tikzcd} \rightsquigarrow \begin{tikzcd} x_1 \arrow[r, "f_1 \circ^h f_2"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A \diamond^v B"] & x_3 \arrow[d, "e_3"] \\ y_1 \arrow[r, "g_1 \circ^h g_2"'] & y_3 \end{tikzcd}$$

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for all x_i,y_i,z_i in $\mathbf{C}^o$, f_i in $\mathbf{C}^h$, e_i,e'_i in $\mathbf{C}^v$ and A, A',B in $\mathbf{C}^s$.

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- Horizontal $(n+1)$-category: category of rewritings, vertical $(n+1)$-category: category of modulo rules.

Polygraphs

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 & & & & & P_n^\top
\end{tikzcd}
$$

- An $(n-1)$-category $\mathcal{C}$ is presented by an n-polygraph $(P_0, \dots, P_n)$ if

$$\mathcal{C} \simeq P_{n-1}^* / \equiv_{P_n}$$

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P^v_{n+\mathbf{1}} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & P^*_n \arrow[l, shift left] \arrow[l, shift right] & P^h_{n+\mathbf{1}} \arrow[u] \arrow[l, shift left] \arrow[l, shift right]
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$$
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 & P^s \arrow[dl, shift right] \arrow[dl, shift left] \arrow[dr, shift right] \arrow[dr, shift left] & \\
(P^v_{n+1})^* \arrow[dr, shift right] \arrow[dr, shift left] & & (P^h_{n+1})^* \arrow[dl, shift right] \arrow[dl, shift left] \\
P^v_{n+1} \arrow[u] \arrow[r, shift right] \arrow[r, shift left] & P^*_n & P^h_{n+1} \arrow[u] \arrow[l, shift right] \arrow[l, shift left]
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$$

- A double $(n+2, n)$-polygraph is a double n-polygraph in which P^s is defined on $((P^v)^\top, (P^h)^\top)$.

- A double $(n+2, n)$-polygraph (P^v, P^h, P^s) generates a free $(n-1)$-category enriched in double groupoids, denoted by $(P^v, P^h, P^s)^{\top\!\!\top}$.

Acyclicity

- A 2-square extension P^s of $((P^v)^\top, (P^h)^\top)$ is acyclic if for any square

$$S = \begin{tikzcd} \cdot \arrow[r, "(P^h)^\top"] \arrow[d, "(P^v)^\top"'] & \cdot \arrow[d, "(P^v)^\top"] \\ \cdot \arrow[r, "(P^h)^\top"'] & \cdot \end{tikzcd}$$

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there exists a square $(n+1)$-cell A in $(P^v, P^h, P^s)^{\top\!\top}$ such that $\partial(A) = S$.

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  there exists a square $(n+1)$-cell A in $(P^v, P^h, P^s)^{\top\!\!\top}$ such that $\partial(A) = S$.

- A 2-fold coherent presentation of an n-category $\mathbf{C}$ is a double $(n+2, n)$-polygraph (P^v, P^h, P^s) such that:
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- From Squier's theorem, $(E, \emptyset, \mathrm{Cd}(E))$ is a 2-fold coherent presentation of $\mathbf{C}$.

III. Polygraphs modulo

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$$\gamma^{{}_E R_E} : {}_E R_E \to \mathrm{Sph}_{n-1}(R^*_{n-1})$$

where ${}_E R_E$ is the set of triples (e, f, e') in $E^\top \times R^{*(1)} \times E^\top$ such that

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and the map $\gamma^{{}_E R_E}$ is defined by $\gamma^{{}_E R_E}(e, f, e') = (\partial_{-,n-1}(e), \partial_{+,n-1}(e'))$.

Branchings and confluence modulo

- A branching modulo E of the n-polygraph modulo S is a triple (f, e, g) where f and g are in S_n^* and e is in $E_n^\top$, such that:

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- Confluence modulo E (resp. local confluence modulo E): any branching (resp. local branching) of S modulo E is confluent modulo E.

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- We consider Γ a 2-square extension of $(E^\top, S^*)$.
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u \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow A"] & u' \arrow[r, dotted, "f'"] & w \arrow[d, dotted, "e'"] \\
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$$\begin{tikzcd}
u \star_i v \arrow[r, "f \star_i v"] \arrow[d, "u \star_i e"'] & u' \star_i v \arrow[d, "u' \star_i e"] \\
u \star_i v' \arrow[r, "f \star_i v'"'] & u' \star_i v'
\end{tikzcd}
\qquad
\begin{tikzcd}
w \star_i u \arrow[r, "w \star_i f"] \arrow[d, "e' \star_i u"'] & w \star_i u' \arrow[d, "e' \star_i u'"] \\
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  - $\mathrm{E} \rtimes \Gamma$ is to avoid "redundant" elements in Γ for different squares corresponding to the same branching of S modulo E:

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, "f'"] & v' \arrow[d, "e'"] \\
u' \arrow[r, "g = e_1 g_1 e_2"'] & w \arrow[r, dotted, "g'"'] & w'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e \star_{n-1} e_1"'] & v \arrow[r, dotted, "f'"] & v' \arrow[d, "e'"] \\
u_1 \arrow[r, "g_1 e_2"'] & w \arrow[r, "g'"'] & w'
\end{tikzcd}$$

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- A set X of $(n-1)$-cells in R^*_{n-1} is E-normalizing with respect to S if for any u in X,

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- **Theorem.** [D.-Malbos '18] Let (R, E, S) be an n-polygraph modulo, and Γ be a square extension of $(E^\top, S^*)$ such that
  - E is convergent,
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  - $\mathrm{Irr}(E)$ is E-normalizing with respect to S,
  - ${}_E R_E$ is terminating,

  then $E \rtimes \Gamma \cup \mathrm{Peiff}(E, S) \cup \mathrm{Cd}(E)$ is acyclic.

Coherent completion

- Coherent completion modulo E of S: square extension of $(E^\top, S^\top)$ containing square cells $A_{f,g}$ and $B_{f,e}$:

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- **Corollary.** [D.-Malbos '18] Let (R, E, S) be an n-polygraph modulo such that
  - E is convergent,
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- **Corollary:** Usual Squier's theorem. $(E = \emptyset)$

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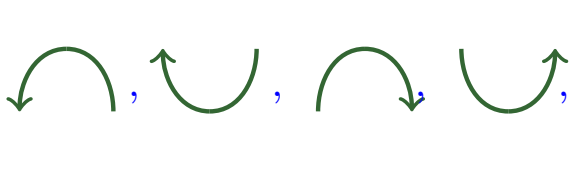
  - $E_2 = \{$  $\}$

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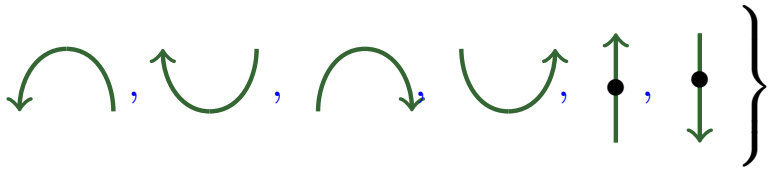
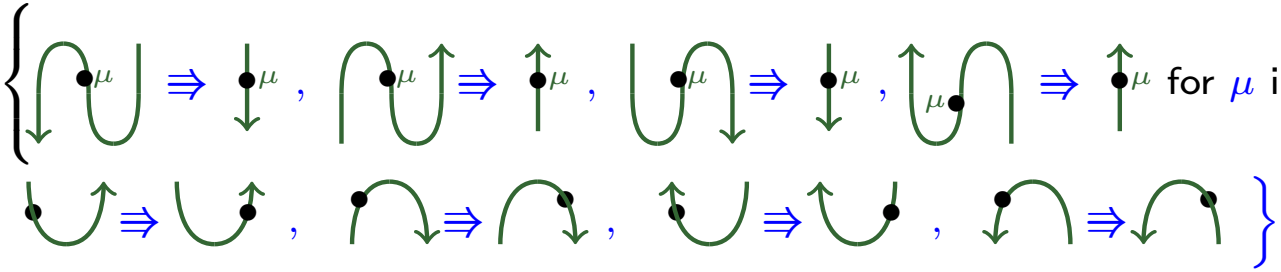
  - $E_2 =$ 
  - $E_3 =$  for μ in $\{0, 1\}$,

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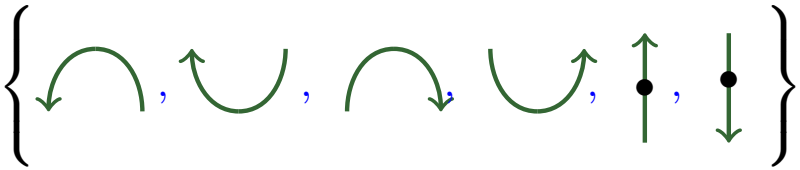
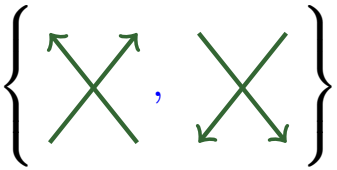
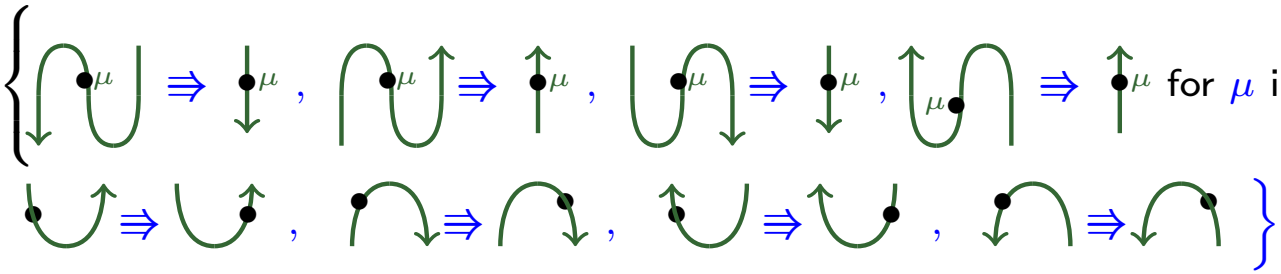
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,

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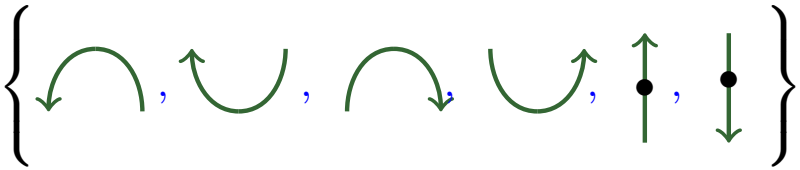
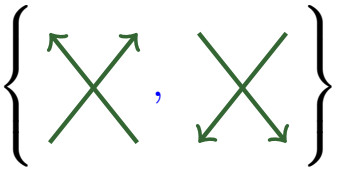
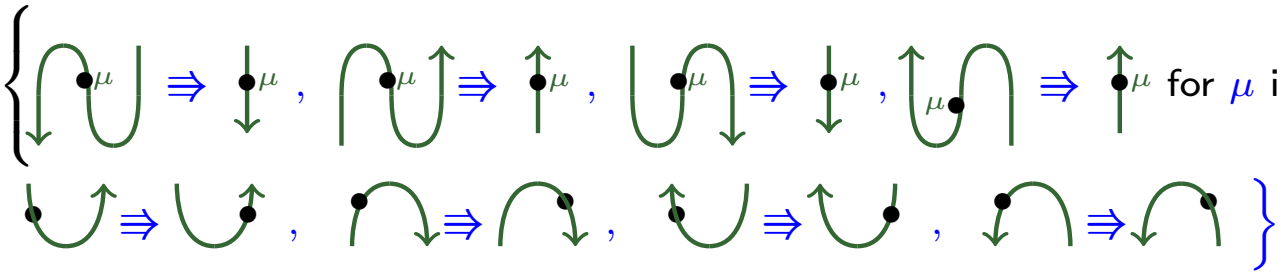
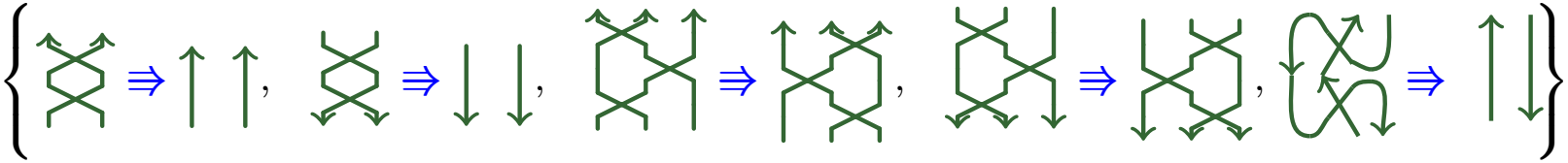
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,
  - $R_3 =$ 

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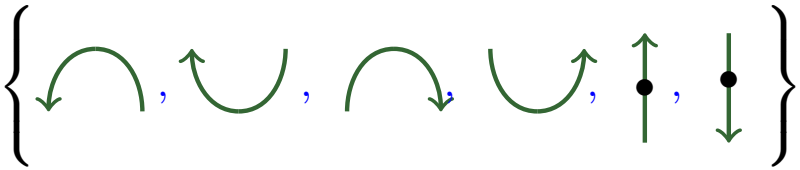
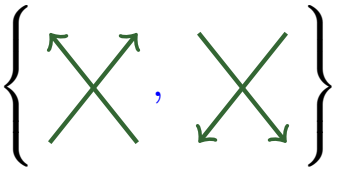
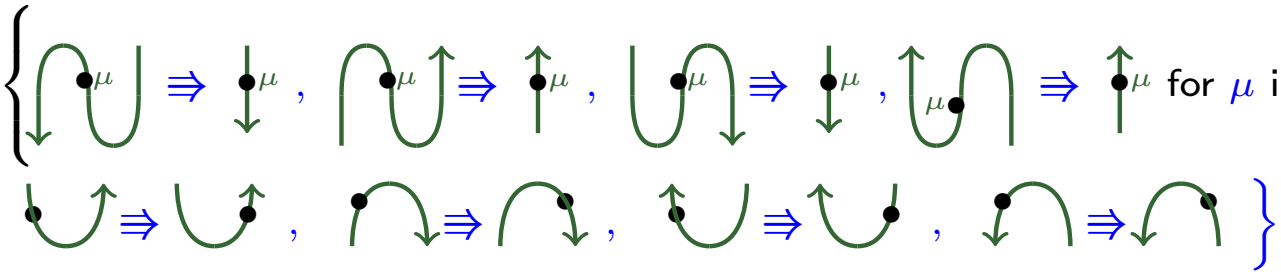
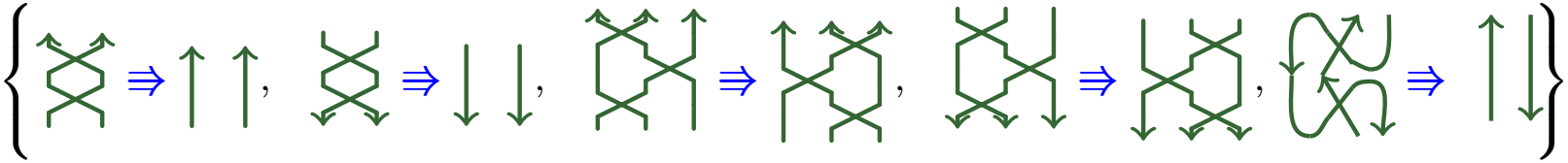
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,
  - $R_3 =$ 
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  - Formalize these constructions with rewriting modulo all the algebraic axioms.

Thank you !