Diagrammatic rewriting modulo isotopy

Benjamin DUPONT, *bdupont@math.univ-lyon1.fr*
Université Claude Bernard Lyon 1, CNRS UMR 5208,
Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne cedex, France

1. INTRODUCTION

This work is part of a research project aiming at developing rewriting methods to study diagrammatic algebras. These diagrammatic algebras appear in various domains of mathematics and physics, as for instance Temperley-Lieb algebras [22] in statistical mechanics, Brauer algebras [3] in representation theory or Birman-Wenzl algebras [21] and Jones' planar algebras [13] in knot theory. Moreover, in representation theory, a new approach has emerged with the idea of studying categorifications of algebras, that is actions of algebras on higher dimensional categories. In this process, some new diagrammatic algebras with a categorical structure appear, such as KLR algebras [16, 20] or Khovanov's diagram algebras [5], and one of the main issue is to compute linear bases of these algebras. Diagrammatic calculus also appear in many other areas of mathematics such as quantum calculus, see for instance [2, 19]. The categories studied in this paper are presented by higher dimensional rewriting systems called 3-polygraphs [10], and rewriting theory provides new methods to study these diagrammatic presentations using termination and confluence properties.

The main issue with equational presentations of these diagrammatic algebras is that there is a huge number of relations, leading to a combinatorial explosion to prove termination and compute critical branchings, and in particular when these algebras can be interpreted as linear categories with additional structure of a pivotal category in which all the diagrams are drawn up by isotopies. For instance, a well-known example is provided by the relation (4) appearing in the Khovanov-Lauda 2-category categorifying a quantum group [17] which is a relation leading to an obstruction for confluence if we consider isotopies as rewriting rules. To deal with this problem, we introduce a context of rewriting modulo isotopy for studying presentations by generators and relations of pivotal 2-categories. This is part of a work aiming at developing rewriting theories in any algebraic structure using rewriting modulo the axioms of the algebraic structure. In this work, I will present a method allowing to obtain coherent presentations modulo mimicking the usual Squier's completion method [11] in a non-modulo setting. In [8], a proof of this result was given using a set theoretical construction for the compositions of the elements of the homotopy basis. In this work, we introduce the notion of 3-dimensional rewriting systems modulo and of $(4,2)$-fold polygraph generating 2-categories enriched in double groupoids, in which the elements of the homotopy bases modulo live. In these double groupoids, the horizontal category deals for the rewritings and the vertical category encodes the modulo. The detailed construction of the free 2-categories enriched in double groupoids generated by a $(4,2)$-fold polygraph and the proof of the main result (Theorem 4.7) can be found in a forecoming paper [9]. We illustrate this result on an exemple of a pivotal 2-category in which the coherent presentation modulo isotopy corresponds to the usual non-modulo coherent presentation.

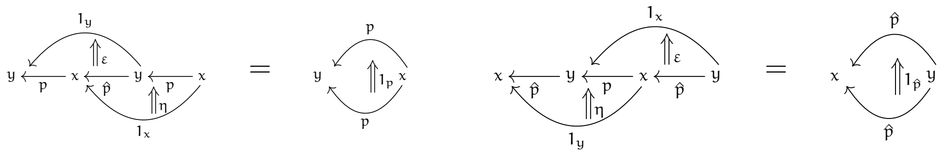
2. REWRITING IN PIVOTAL 2-CATEGORIES

Computation of coherent presentations for diagrammatic algebras may lead to a combinatorial explosion due to the computation of critical pairs. In particular, we illustrate this problem with the case of diagrammatic algebras that can be interpreted as pivotal 2-categories, that is 2-categories in which all 2-cells are drawn up to isotopy. These algebras arise in many situations in representation theory. We motivate the use of rewriting modulo to study presentations by generators and relations of these pivotal 2-categories.

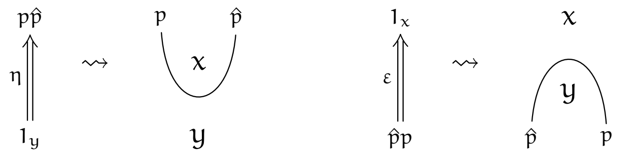
2.1. Adjunctions in a 2-category. Let $\mathcal{C}$ be a 2-category with sets of 0-cells, 1-cells and 2-cells respectively denoted by $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$. For any 1-cell p in $\mathcal{C}_1$, a *right adjoint* of p is a 1-cell $\hat{p} : y \to x$ equipped with two 2-cells ε and η in $\mathcal{C}$ defined as follows:

$$
\begin{tikzcd}
y & x \arrow[l, "\hat{p}"'] & y \arrow[l, "p"'] \arrow[ll, bend right=60, "1_y"'] \arrow[l, phantom, shift right=18, "\Uparrow\varepsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x & y \arrow[l, "\hat{p}"'] & x \arrow[l, "p"'] \arrow[ll, bend left=60, "1_x"] \arrow[l, phantom, shift left=18, "\Uparrow\eta"]
\end{tikzcd}
,
$$

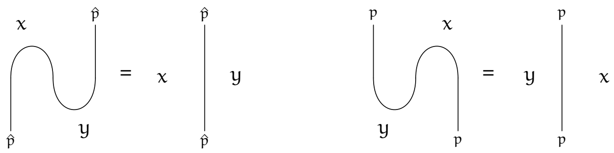
called the *counit* and *unit* of the adjunction, such that the equalities



hold. We denote the fact that p is a left adjoint of $\hat{p}$ by $p \dashv \hat{p}$. In a string diagrammatic notation, these units and counits are represented by caps and cups as follows:



The axioms of an adjunction require that the equalities between composites of 2-morphisms


(1)

are satisfied. When we are in the situation where $\hat{p}$ is also a left-adjoint of p, that is p and $\hat{p}$ are biadjoint, that we denote by $p \dashv \hat{p} \dashv p$, the symmetric zig-zag relations hold similarly.

2.2. Mateship under adjunction. We recall following [18] the 2-category theoretic notion of mateship under adjunction introduced by Kelly and Street in [15]. This is a certain correspondence between 2-cells in the presence of adjoints.

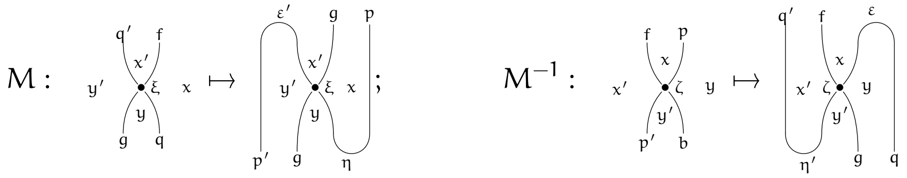
Given adjoints $\eta, \varepsilon \colon p \dashv q \colon x \to y$ and $\eta', \varepsilon' \colon p' \dashv q' \colon x' \to y'$ in the 2-category $\mathcal{C}$, for any 1-cells $f \colon x \to x'$ and $g \colon y \to y'$ in $\mathcal{C}$, there is a bijection M between 2-cells $\xi \in \mathcal{C}(g \star_0 q, q' \star_0 f)$ and 2-cells $\zeta \in \mathcal{C}(p' \star_0 g, f \star_0 p)$, where ζ is given in terms of ξ by the composite:

$$
\begin{aligned}
M \colon \mathcal{C}(g \star_0 q, q' \star_0 f) \quad &\longrightarrow \quad \mathcal{C}(p' \star_0 g, f \star_0 p) \\
\xi \quad &\mapsto \quad \left(p' \star_0 g \xRightarrow{p' \star_0 g\eta} p' \star_0 g \star_0 q \star_0 p \xRightarrow{p' \star_0 \xi \star_0 p} p' \star_0 q' \star_0 f \star_0 p \xRightarrow{\varepsilon' \star_0 f \star_0 p} f \star_0 p \right) = \zeta\,,
\end{aligned}
$$

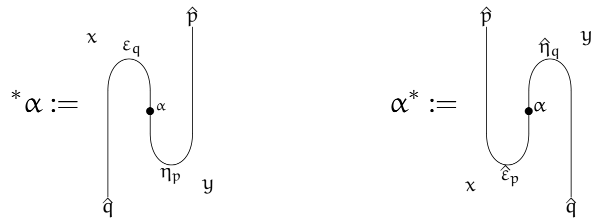
and ξ is given in terms of ζ by the composite:

$$
\begin{aligned}
M^{-1} \colon \mathcal{C}(p' \star_0 g, f \star_0 p) \quad &\longrightarrow \quad \mathcal{C}(g \star_0 q, q' \star_0 f) \\
\zeta \quad &\mapsto \quad \left(g \star_0 q \xRightarrow{\eta' \star_0 g \star_0 q} q' \star_0 p' \star_0 g \star_0 q \xRightarrow{q' \star_0 \zeta \star_0 q} q' \star_0 f \star_0 p \star_0 q \xRightarrow{q' \star_0 f \star_0 \varepsilon} q' \star_0 f \right) = \xi\,.
\end{aligned}
$$

We then say that ξ and ζ are *mates under adjunction*. Diagrammatically, this notion of mateship under adjunction can be expressed as:



2.3. Cyclic 2-cells and pivotality. Given a pair of 1-cells $p, q\colon x \to y$ with chosen biadjoints $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$, then any 2-cell $\alpha\colon p \Rightarrow q$ has two obvious duals ${}^*\alpha, \alpha^*\colon \hat{q} \Rightarrow \hat{p}$, or mates, one constructed using the left adjoint structure, the other using the right adjoint structure. Diagrammatically the two mates are given by



(2)

We will call α^* the right dual of α because it is obtained from α as its mate using the right adjoints of p and q. Similarly, ${}^*\alpha$ is called the left dual of α because it is obtained from α as its mate using the left adjoints of p and q.

In general there is no reason why ${}^*\alpha$ should be equal to α^*, see [18] for a simple counterexample.

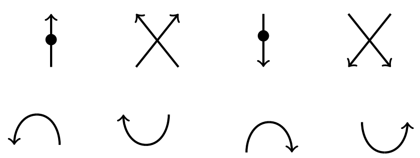
2.4 Definition ([6]). Given biadjoints $(p, \hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(q, \hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$ and a 2-cell $\alpha\colon p \Rightarrow q$ define $\alpha^* := \hat{p}\hat{\eta}_q.\hat{\varepsilon}_p\hat{q}$ and ${}^*\alpha := \varepsilon_G\hat{p}.\hat{\eta}_q$ as above. Then a 2-cell α is called a *cyclic 2-cell* if the equation ${}^*\alpha = \alpha^*$ is satisfied, or either of the equivalent conditions ${}^{**}\alpha = \alpha$ or $\alpha^{**} = \alpha$ are satisfied.

A 2-category in which all the 2-cells are cyclic with respect to some biadjunction is called a *pivotal* 2-category. In this categorical structure, on gets that all string diagram representing 2-cells are drawn up by isotopy from the following theorem:

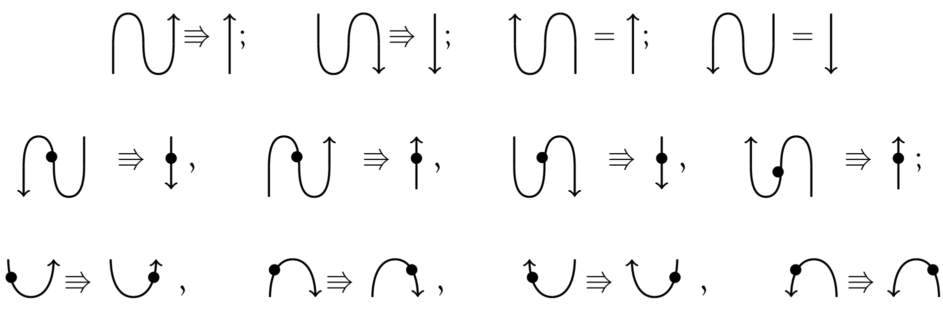
2.5 Theorem ([6]). *Given a string diagram representing a cyclic 2-cell, between 1-cells with chosen biadjoints, then any isotopy of the diagram represents the same 2-cell.*

2.6. Example. We consider a 3-polygraph Σ with

- only one 0-cell $*$;
- two 1-cells $\wedge$ and $\vee$;
- the 2-cells of Σ are defined by

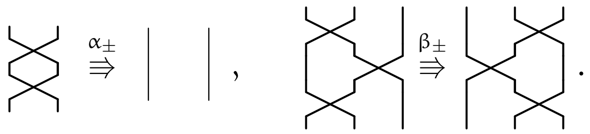


- the 3-cells of Σ are given by:
  - the 3-cells of the polygraphs of pearls from [10], making $\wedge$ and $\vee$ biadjoints and $\bullet$ a cyclic 2-cell:

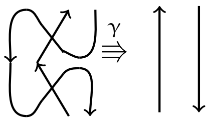


(3)

– the 3-cells of the 3-polygraph of permutations for both upward and downward orientations of strands:



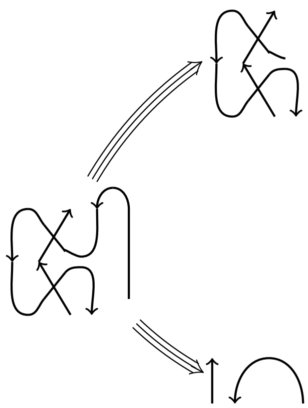
– an additional 3-cell



(4)

which is a well-known example of relation arising in representation theory, see for instance Khovanov-Lauda's 2-category [17] which categorifies quantum groups associated with symmetrizable Kac-Moody algebras.

One can check that this 3-polygraph is not confluent since the branching



(5)

is not confluent. However, the isotopy relations are part of the algebraic structure, and should not be considered as rewriting rules for the study of confluence. To solve this problem, we introduce a concept of *rewriting modulo isotopy*.

3. Rewriting modulo

As we have just seen, the main objective is to handle confluence obstructions coming from relations inherent to the algebraic structure, such for instance isotopies in pivotal 2-categories. We introduce the notion of rewriting system modulo and enounce a critical pair lemma in that context.

3.1. 3-dimensional rewriting systems modulo. A *3-dimensional rewriting system modulo* is the data of a quadruple $(\mathsf{P}, \mathsf{R}, \mathsf{E}, \mathsf{S})$ where:

- P is a 1-polygraph;
- R is a 3-polygraph admitting P as underlying 1-polygraph;
- E is a 3-polygraph admitting P as underlying 1-polygraph and such that $\mathsf{E}_2 \subseteq \mathsf{P}_2$;
- S is a cellular extension of R_2^* which depend on E in the following sense: any element of S is a 3-cell in $\mathsf{P}_2^*(\mathsf{E})[\mathsf{R}]$ the *free 2-category of rewritings with respect to* R *modulo* E, which is defined by:
$$\mathsf{P}_2^*(\mathsf{E})[\mathsf{R}] = (\mathsf{E} \cup \overline{\mathsf{E}} \cup \mathsf{R})^*/\text{Inverse}(\mathsf{E}, \overline{\mathsf{E}})$$
where $\overline{\mathsf{E}}$ is defined by the set of formal inverses of elements of E and Inverse$(\mathsf{E}, \overline{\mathsf{E}})$ is given by relations $\mathsf{f} \star_2 \overline{\mathsf{f}} = 1_{\mathsf{s}_1(\mathsf{f})}$ and $\overline{\mathsf{f}} \star_2 \mathsf{f} = 1_{\mathsf{t}_1(\mathsf{f})}$ for any 2-cell f in E^*.

We denote by $E^\top$ the free $(3,2)$-category generated by E, and we say that two 2-cells u and v in P_2^* are E-equivalent if they are linked by a 3-cell of $E^\top$, and we denote this by $u \approx_E v$. In the literature [12, 14, 1], a 3-dimensional rewriting system modulo (P, R, E, S) corresponds to a 3-dimensional rewriting system S satisfying $R \subseteq S \subseteq {}_E R_E$, where ${}_E R_E$ is the rewriting system defined by rewriting on E-equivalence classes, that is rules $u \Rrightarrow v$ iff there exists u' and v' in P_2^* such that $u' \approx_E u$, $v' \approx_E v$ and $u' \Rrightarrow v'$ in R.

3.2. Critical pair lemma modulo. A *branching modulo* E for a 3-polygraph S is a pair (f, g) of 3-cells of S^* such that $s_1(f) \approx_E s_1(g)$ as depicted by

$$\begin{tikzcd} & u' \\ u \arrow[ur, Rrightarrow, "f"] \arrow[d, Rrightarrow, "e"] & \\ v \arrow[dr, Rrightarrow, "g"'] & \\ & v' \end{tikzcd} \qquad (6)$$

Following [9, 14], for a terminating 3-dimensional rewriting system S modulo E, the property of confluence modulo E is equivalent to the confluence of the critical branchings modulo of the following form:

$$\begin{tikzcd} & u' & \\ u \arrow[ur, Rrightarrow, "f"] \arrow[dr, Rrightarrow, "g"'] & & w \\ & v & \end{tikzcd} \qquad \begin{tikzcd} & u' & \\ u \arrow[ur, Rrightarrow, "f"] \arrow[dr, Rrightarrow, dash, "e"'] & & w \\ & v & \end{tikzcd}$$

for any 2-cells f in S^*, g in R^* and e in $E^\top$ each of length 1.

4. Coherence modulo

Let us introduce all the categorical material needed to enounce the coherence theorem modulo. Then, we illustrate this result on the example above to obtain an homotopy basis modulo isotopy.

4.1. Double groupoids . We denote by Cat the category of all (small) categories and functors. A *groupoid* is a category $\mathcal{G}$ whose any 1-cell is invertible. We denote by Grpd the category of groupoids and functors.

A *double category* (resp. *double groupoid*) is an internal category $(\mathcal{C}_1, \mathcal{C}_0, \partial_-^{\mathcal{C}}, \partial_+^{\mathcal{C}}, \circ_{\mathcal{C}}, i_{\mathcal{C}})$ in Cat (resp. Grpd). A double category (resp. double groupoid) $\mathcal{C}$ gives four related categories (resp. groupoids)

$$\mathcal{C}^{sh} := (\mathcal{C}^s, \mathcal{C}^h, \partial^h_{-,1}, \partial^h_{+,1}, \diamond^h, i^h_1), \qquad \mathcal{C}^{sv} := (\mathcal{C}^s, \mathcal{C}^v, \partial^v_{-,1}, \partial^v_{+,1}, \diamond^v, i^v_1),$$
$$\mathcal{C}^{ho} := (\mathcal{C}^h, \mathcal{C}^o, \partial^h_{-,0}, \partial^h_{+,0}, \circ^h i^h_0), \qquad \mathcal{C}^{vo} := (\mathcal{C}^v, \mathcal{C}^o, \partial^v_{-,0}, \partial^v_{+,0}, \circ^v, i^v_0).$$

where $\mathcal{C}^{sh}$ is the category $\mathcal{C}_1$ and $\mathcal{C}^{vo}$ is the category $\mathcal{C}_0$. The sources and target maps

$$\begin{tikzcd} & \mathcal{C}^s \arrow[dl, shift left, "\partial^h_{+,1}"'] \arrow[dl, shift right, "\partial^h_{-,1}"] \arrow[dr, shift left, "\partial^v_{+,1}"] \arrow[dr, shift right, "\partial^v_{-,1}"'] & \\ \mathcal{C}^h \arrow[dr, shift left, "\partial^h_{+,0}"] \arrow[dr, shift right, "\partial^h_{-,0}"'] & & \mathcal{C}^v \arrow[dl, shift left, "\partial^v_{-,0}"] \arrow[dl, shift right, "\partial^v_{+,0}"'] \\ & \mathcal{C}^o & \end{tikzcd}$$

satisfies the following relations:

i) $\partial^h_{\alpha,0}\partial^h_{\beta,1} = \partial^v_{\beta,0}\partial^v_{\alpha,1}$ for all α, β in $\{-,+\}$,

ii) $\partial^v_{\alpha,0} i^h_1 = i_0 \partial^h_{\alpha,0}$, $\partial^h_{\alpha,0} i^v_1 = i^h_0 \partial^v_{\alpha,0}$, for all α in $\{-,+\}$,

iii) $i^v_1 i^v_0 = i^h_1 i^h_0$,

iv) $\partial^\mu_{\alpha,1}(A \diamond^\mu B) = \partial^\mu_{\alpha,1}(A) \circ^\mu \partial^\mu_{\alpha,1}(B)$, for all $\alpha \in \{-,+\}$, $\mu \in \{v, h\}$ and any squares A, B such that both sides are defined.

v) exchange law : $(A \diamond^v A') \diamond^h (B \diamond^h B') = (A \diamond^h B) \diamond^v (A' \diamond^h B')$, for any squares A, A', B, B' such that both sides are defined.

Elements of $\mathcal{C}^h$ and $\mathcal{C}^v$ and $\mathcal{C}^s$ are respectively called *horizontal* arrows, *vertical* arrows and *squares* and can be pictured as follows:

$$
\begin{tikzcd} x_1 \arrow[r, "f"] & x_2 \end{tikzcd}
\qquad
\begin{tikzcd} x_1 \arrow[d, "e"'] \\ x_2 \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\partial^h_{-,1}(A)"] \arrow[d, "\partial^v_{-,1}(A)"'] \arrow[dr, phantom, "\Downarrow A"] & \cdot \arrow[d, "\partial^v_{+,1}(A)"] \\
\cdot \arrow[r, "\partial^h_{+,1}(A)"'] & \cdot
\end{tikzcd}
$$

Compositions in such a double groupoid are defined as follows:

$$
\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1"'] & y_2 \arrow[r, "g_2"'] & y_3
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}[column sep=huge]
x_1 \arrow[r, "f_1 \circ^h f_2"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A \diamond^h B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1 \circ^h g_2"'] & y_3
\end{tikzcd}
$$

for all x_i, y_i in $\mathcal{C}^o$, f_i, g_i in $\mathcal{C}^h$, e_i in $\mathcal{C}^v$ and A, B in $\mathcal{C}^s$,

$$
\begin{tikzcd}
x_1 \arrow[r, "f"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] \\
y_1 \arrow[r, "g"'] \arrow[d, "e_1'"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e_2'"] \\
z_1 \arrow[r, "h"'] & z_2
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}[row sep=huge]
x_1 \arrow[r, "f"] \arrow[d, "e_1 \circ^v e_1'"'] \arrow[dr, phantom, "\Downarrow A \diamond^v A'"] & x_2 \arrow[d, "e_2 \circ^v e_2'"] \\
z_1 \arrow[r, "h"'] & z_2
\end{tikzcd}
$$

for all x_i, y_i, z_i in $\mathcal{C}^o$, f, g, h in $\mathcal{C}^h$, e_i, e_i' in $\mathcal{C}^v$ and A, A' in $\mathcal{C}^s$.

4.2. Square extensions. Given two 3-polygraphs S and E with the same underlying 1-polygraph P, a 2-*square extension* of the pair of 3-categories $(S^\top, E^\top)$ is a set S^s_4 equipped with four maps $\partial^v_{-,1}, \partial^v_{+,1} : S^s_4 \to E^\top_3$ and $\partial^h_{-,1}, \partial^h_{+,1} : S^s_4 \to S^\top_3$ satisfying the following cubical relations making S^s_4 a 2-cubical set in the sense of [4]:

$$\partial^h_{\alpha,0}\partial^h_{\beta,1} = \partial^v_{\beta,0}\partial^v_{\alpha,1} \text{ for all } \alpha, \beta \text{ in } \{-,+\}.$$

An element of S^s_4 is called a 2-*square* in $(S^\top, E^\top)$, and is depicted as follows

$$
\begin{tikzcd}
\cdot \arrow[r, Rightarrow, "f_1"] \arrow[d, Rightarrow, "e_1"'] & \cdot \arrow[d, Rightarrow, "e_2"] \\
\cdot \arrow[r, Rightarrow, "f_2"'] & \cdot
\end{tikzcd}
$$

with $f_i \in S^\top$ and $e_i \in E^\top$.

4.3. $(4,2)$-fold polygraphs. We introduce the notion of $(4,2)$-fold polygraph, which is the structure corresponding to a coherent presentation modulo of two 3-polygraphs E and R as in 3.1. The number 4 deals for the dimension of rewriting, and $2 = 4 - 2$ encodes for the enrichment in 2-fold categories (double categories) in a sense that we will explicit.

A $(4,2)$-*fold polygraph* is the data of a tuple (P, E, S, Γ) where:

- S a 3-dimensional rewriting system modulo and E a 3-polygraph with the same underlying 1-polygraph P, and $E_2 \subseteq S_2$;
- Γ is a 2-square extension of $(S^\top, E^\top)$.

This corresponds to the following diagram:

$$\begin{tikzcd}
 & & \Gamma \arrow[dll, shift right, "\partial^h_{+,1}"'] \arrow[dll, shift left, "\partial^h_{-,1}"] \arrow[drr, shift left, "\partial^v_{-,1}"] \arrow[drr, shift right, "\partial^v_{+,1}"'] & & \\
S_3^\top \arrow[drr, shift right, "\partial^v_{+,0}"'] \arrow[drr, shift left, "\partial^v_{-,0}"] & & & & E_3^\top \arrow[dll, shift left, "\partial^h_{-,0}"] \arrow[dll, shift right, "\partial^v_{+,0}"'] \\
S_3 \arrow[u] \arrow[rr, shift left] \arrow[rr, shift right] & & S_2^* \arrow[d, shift right, "\partial_{+,1}"'] \arrow[d, shift left, "\partial_{-,1}"] & & E_3 \arrow[u] \arrow[ll, shift left] \arrow[ll, shift right] \\
 & & P_1^* & &
\end{tikzcd}$$

4.4. Free 2-category enriched in double groupoids generated by a $(4,2)$-fold polygraph. We recall that a 2-category enriched in double groupoids is a 2-category $\mathcal{C}$ such that for any p, q in $\mathcal{C}_1$ the homset $\mathcal{C}(p,q)$ has a double groupoid structure, where the set $\mathcal{C}(p,q)^o$ is the set of 2-cells in $\mathcal{C}_1(p,q)$. We will denote by $\mathcal{C}_3^v$ (resp. $\mathcal{C}_3^h$, $\mathcal{C}_4^s$) the sets $\mathcal{C}(p,q)^v$ (resp. $\mathcal{C}(p,q)^h$, $\mathcal{C}(p,q)^s$) for all p, q in $\mathcal{C}_1$. A 4-cell A in $\mathcal{C}_4^s$ can be represented by the following diagrams:

$$\begin{tikzcd}
u \arrow[r, Rightarrow, "f"] \arrow[d, Rightarrow, "e"'] \arrow[dr, phantom, "\Downarrow A"] & v \arrow[d, Rightarrow, "e'"] \\
u' \arrow[r, Rightarrow, "g"'] & v'
\end{tikzcd}$$

with $u, u', v, v' \in \mathcal{C}_2(p,q)$, $f, g \in \mathcal{C}_2^h$ and $e, e' \in \mathcal{C}_2^v$.

The composition $\star_1$ of 4-cells along 1-cells is induced by the functor of double categories

$$\star_1^{p,q,r} : \mathcal{C}_1(p,q) \times \mathcal{C}_1(q,r) \to \mathcal{C}_1(p,r)$$

for any 1-cells p, q, r, and are denoted as follows: the 1-composite of a 4-cell A in $\mathcal{C}(p,q)$ with a 4-cell B in $\mathcal{C}(q,r)$ such that

$$\begin{tikzcd}
u_1 \arrow[r, Rightarrow, "f_1"] \arrow[d, Rightarrow, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & v_1 \arrow[d, Rightarrow, "e_1'"] \\
u_1' \arrow[r, Rightarrow, "g_1"'] & v_1'
\end{tikzcd}
\qquad
\begin{tikzcd}
u_2 \arrow[r, Rightarrow, "f_2"] \arrow[d, Rightarrow, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & v_2 \arrow[d, Rightarrow, "e_2'"] \\
u_2' \arrow[r, Rightarrow, "g_2"'] & v_2'
\end{tikzcd}$$

with u_1, u_1', v_1, v_1' in $\mathcal{C}_2(p,q)$,

$$\begin{tikzcd}
u_1 \star_1 u_2 \arrow[r, Rightarrow, "f_1\star_1 f_2"] \arrow[d, Rightarrow, "e_1\star_1 e_2"'] \arrow[dr, phantom, "\Downarrow A\star_1 B"] & v_1 \star_1 v_2 \arrow[d, Rightarrow, "e_1'\star_1 e_2'"] \\
u_1' \star_1 u_2' \arrow[r, Rightarrow, "g_1\star_1 g_2"'] & v_1' \star_1 v_2'
\end{tikzcd}$$

The other compositions in $\mathcal{C}(p,q)$ are given by the vertical and horizontal compositions of the double groupoid as in 4.1.

Given a $(4,2)$-fold polygraph $\Sigma = (P, E, S, \Gamma)$, we denote by $\Sigma^{\top\!\!\top}$ the free 2-category enriched in double groupoids generated by Σ. Its underlying 2-category is the free 2-category S_2^*, and for any 1-cells p and q in P_1^*, we define a double groupoid $S^{\partial}(x,y)$ by the following diagram:

$$\begin{tikzcd}
 & \Gamma^{\top} \arrow[dl, shift right, "\partial^h_{+,1}"'] \arrow[dl, shift left, "\partial^h_{-,1}"] \arrow[dr, shift left, "\partial^v_{+,1}"] \arrow[dr, shift right, "\partial^v_{-,1}"'] & \\
S_3^{\top} \arrow[dr, shift left, "\partial^h_{+,0}"] \arrow[dr, shift right, "\partial^h_{-,0}"'] & & E_3^{\top} \arrow[dl, shift right, "\partial^v_{+,0}"'] \arrow[dl, shift left, "\partial^v_{-,0}"] \\
 & S_2^* &
\end{tikzcd}$$

where $\partial^h_{\pm,0}$ and $\partial^v_{\pm,0}$ correspond to 2-source and 2-target maps of the free 3-categories $S^{\top}$ and $E^{\top}$, and $\Gamma^{\top}$ is obtained by $\star_0$, $\star_1$-compositions of any 4-cells A in $S^{\partial}(x,y)$ and B in $S^{\partial(y,z)}$ and their formal inverses defined fonctorially as above, and vertical and horizontal compositions of 4-cells and their formal inverses in each double groupoid $S^{\partial(x,y)}$.

4.5. Acyclicity. Let S be a 3-dimensional rewriting system and E be a 3-polygraph with the same underlying 1-polygraph P. A 2-square extension Γ of $(S^{\top}, E^{\top})$ is said *acyclic* if for any 2-square

$$\begin{tikzcd}
u \arrow[r, Rightarrow, "f"] \arrow[d, Rightarrow, "e"'] & v \arrow[d, Rightarrow, "e'"] \\
u' \arrow[r, Rightarrow, "g"'] & v'
\end{tikzcd}$$

in $(S^{\top}, E^{\top})$, there exists a 4-cell A in the free 2-category enriched in double groupoids generated by the $(4,2)$-fold polygraph (P, E, S, Γ) such that

$$\begin{tikzcd}
u \arrow[r, Rightarrow, "f"] \arrow[d, Rightarrow, "e"'] \arrow[dr, phantom, "\Downarrow A"] & v \arrow[d, Rightarrow, "e'"] \\
u' \arrow[r, Rightarrow, "g"'] & v'
\end{tikzcd}$$

4.6. Squier's completion modulo. Let (P, R, E, S) be a 3-dimensional rewriting system modulo. A *Squier's completion modulo* E of the 3-dimensional rewriting system S is a 2-square extension of $(S^{\top}, E^{\top})$

whose elements are the 4-cells $A_{f,g}$ or $B_{f,e}$ of the following form:

$$\begin{tikzcd}
& u' \arrow[r, Rightarrow, "f'"] & w \arrow[d, squiggly, "e'"] \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & \Downarrow A_{f,g} & w' \\
& v \arrow[ur, Rightarrow, "g'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& u' \arrow[r, Rightarrow, "f'"] & w \arrow[d, squiggly, "e'"] \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, "e"'] & \Downarrow B_{f,e} & w' \\
& v \arrow[ur, Rightarrow, "g'"'] &
\end{tikzcd}$$

for any critical branching (f, g) and (f, e) of S modulo F, where f, g and e belongs to S^*, R^* and $F^\top$. Note that such completion is not unique in general and depends on the rewriting sequences f', g' and the equivalence e' used to obtain the confluence diagrams. These confluence diagrams corresponds to the following 2-squares in $(E^\top, S^\top)$:

$$\begin{tikzcd}
u \arrow[r, Rightarrow, "f\star_2 f'"] \arrow[d, squiggly, "e"'] \arrow[dr, phantom, "\Downarrow A_{f,g}"] & w \arrow[d, squiggly, "1_u"] \\
u \arrow[r, Rightarrow, "g\star_2 g'"'] & w'
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, Rightarrow, "f\star_2 f'"] \arrow[d, squiggly, "e"'] \arrow[dr, phantom, "\Downarrow B_{f,e}"] & w \arrow[d, squiggly, "e'"] \\
v \arrow[r, Rightarrow, "g"'] & w'
\end{tikzcd}$$

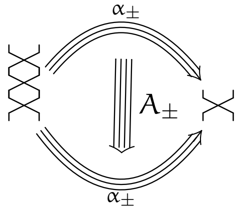
4.7 Theorem ([9]). *Let (P, R, E, S) be a 3-dimensional rewriting system modulo such that S is terminating and confluent modulo E. Then any Squier's completion of S modulo E is an acyclic 2-square extension.*

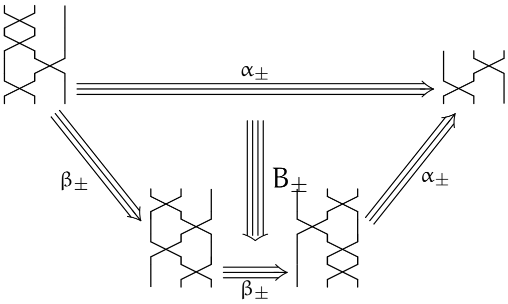
4.8. Example. In this subsection, we illustrate this on a 3-dimensional diagrammatic rewriting system modulo isotopies. We consider the 3-polygraph E which has:

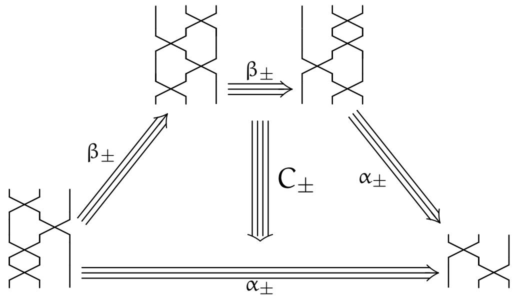
- only one 0-cell $*$;
- two 1-cells $\wedge$ and $\vee$;
- four 2-cells defined by 
- the isotopy 3-cells of the 3-polygraph of pearls (3).

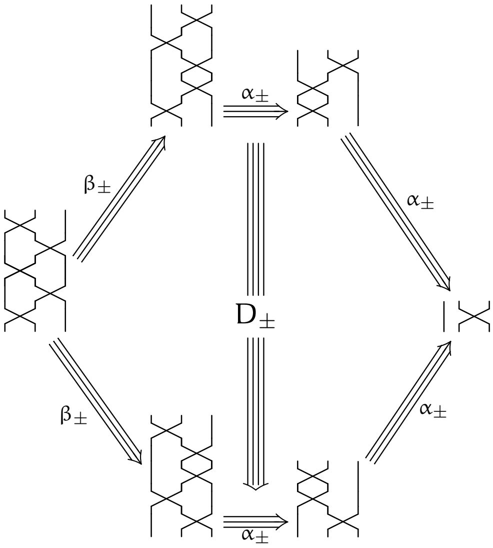
We consider the 3-polygraph R which has the same i-cells than the 3-polygraph Σ of 2.6 for $0 \leq i \leq 2$ and whose set of 3-cells is given by $(\alpha_\pm, \beta_\pm, \gamma)$ and let S be the 3-dimensional rewriting system modulo defined by 3-cells $u \Rrightarrow v$ whenever there exists u' in R_2^* such that $u \approx_E u'$ and $u' \Rrightarrow v$ is a rewriting step in R.

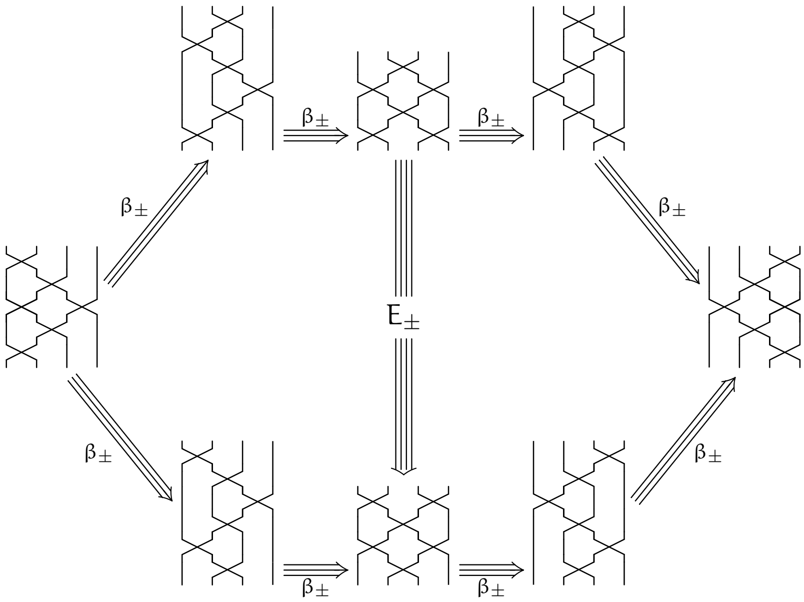
Following [9], the only critical branchings we have to consider in that exemple are the pairs (f, g) for f in S^* and g in R^* both of length 1. Notice that the branching (5) is not such a critical branching because the top 3-cell lives in $E^\top$, and the top-right 2-cell is not reducible by R. More over, one can check that the only critical branchings modulo E in Σ are given by pairs (f, g) of 3-cells both in R^* of length 1. As relation γ does not overlap with any other 3-cell, the only critical branchings we have to consider are those of the 3-polygraph of permutations [10], with either upward or downward orientated strands. This lead to the following elements of the homotopy basis defined for both orientations of strands:











CONCLUSION

In this work, we presented a method allowing to compute coherent presentations using rewriting modulo. We illustrated the main result on an example showing that if some rules such as isotopies (3) are not considered as axioms for which we rewrite modulo, it is hopeless to obtain homotopy bases using convergent presentations for diagrammatic algebras. The works in progress attempt to apply these methods in various algebraic structures, to obtain coherence results for instance for groups, or commutative monoids or algebras.

REFERENCES

[1] Leo Bachmair and Nachum Dershowitz. Completion for rewriting modulo a congruence. *Theoretical Computer Science*, 67(2):173 – 201, 1989.

[2] K. Bar, A. Kissinger, and J. Vicary. Globular: an online proof assistant for higher-dimensional rewriting. *ArXiv e-prints*, December 2016.

[3] Richard Brauer. On algebras which are connected with the semisimple continuous groups. *Annals of Mathematics*, 38(4):857–872, 1937.

[4] Ronald Brown and Philip J. Higgins. On the algebra of cubes. *Journal of Pure and Applied Algebra*, 21(3):233 – 260, 1981.

[5] Jonathan Brundan and Catharina Stroppel. Highest weight categories arising from Khovanov's diagram algebra I: cellularity. *Mosc. Math. J.*, 11(4):685–722, 821–822, 2011.

[6] J. R. B. Cockett, J. Koslowski, and R. A. G. Seely. Introduction to linear bicategories. *Math. Structures Comput. Sci.*, 10(2):165–203, 2000. The Lambek Festschrift: mathematical structures in computer science (Montreal, QC, 1997).

[7] Robert Dawson and Robert Paré. What is a free double category like? *J. Pure Appl. Algebra*, 168(1):19–34, 2002.

[8] Benjamin Dupont and Philippe Malbos. Coherence modulo relation. *International workshop on confluence*, 2018.

[9] Benjamin Dupont and Philippe Malbos. Coherence modulo relations and double groupoids. *Work in progress*, 2018.

[10] Yves Guiraud and Philippe Malbos. Higher-dimensional categories with finite derivation type. *Theory Appl. Categ.*, 22:No. 18, 420–478, 2009.

[11] Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. *Math. Structures Comput. Sci.*, 28(2):155–201, 2018.

[12] Gérard Huet. Confluent reductions: abstract properties and applications to term rewriting systems. *J. Assoc. Comput. Mach.*, 27(4):797–821, 1980.

[13] V. F. R. Jones. Planar algebras, I. *ArXiv Mathematics e-prints*, September 1999.

[14] Jean-Pierre Jouannaud and Helene Kirchner. Completion of a set of rules modulo a set of equations. In *Proceedings of the 11th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages*, POPL '84, pages 83–92, New York, NY, USA, 1984. ACM.

[15] G. M. Kelly and Ross Street. Review of the elements of 2-categories. pages 75–103. Lecture Notes in Math., Vol. 420, 1974.

[16] M. Khovanov and A. D. Lauda. A diagrammatic approach to categorification of quantum groups I. *ArXiv e-prints*, March 2008.

[17] M. Khovanov and A. D. Lauda. A diagrammatic approach to categorification of quantum groups III. *ArXiv e-prints*, July 2008.

[18] Aaron D. Lauda. An introduction to diagrammatic algebra and categorified quantum $\mathrm{sl}(2)$. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 7(2):165–270, 2012.

[19] David Reutter and Jamie Vicary. A 2-Categorical Approach to Composing Quantum Structures. In Filippo Bonchi and Barbara König, editors, *7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017)*, volume 72 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 20:1–20:20, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[20] R. Rouquier. 2-Kac-Moody algebras. *ArXiv e-prints*, December 2008.

[21] Joan S. Birman and Hans Wenzl. Braids, link polynomials and a new algebra. 313:249–249, 05 1989.

[22] H. N. V. Temperley and E. H. Lieb. Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the 'percolation' problem. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 322(1549):251–280, 1971.