A convergent presentation for the simply-laced KLR algebra and the PBW property

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INTRODUCTION

In higher representation theory, we study actions of algebras on categories rather than vector spaces. When we start from an algebra presented by generators and relations, we can study its 2-representations by constructing a categorification of the given algebra, that is an higher dimensional category whose Grothendieck group is isomorphic to this latter. In [KL08a, KL08b, KL08c] they studied for instance some categorifications of the quantum groups $\dot{U}(\mathfrak{g})$ associated with a given Kac-Moody algebra $\mathfrak{g}$ to obtain 2-representations of $\mathfrak{g}$. Such a Kac-Moody algebra is defined from a graph; here we restrict to the case of a simply-laced graph, that is to say a graph without loops and multiple edges. In order to build their 2-categories, Khovanov and Lauda introduced a family of algebras called KLR algebras, or quiver Hecke algebras. In this work, we construct using rewriting methods Poincaré-Birkhoff-Witt (PBW) bases of the KLR algebras. These bases were introduced by Rouquier in [Rou08]. We will apply the rewriting rules on braid-like diagrams which link a sequence of vertices of the graph to another sequence. In [KL08a, Theorem 2.5], Khovanov and Lauda already showed that in the simply-laced case we can construct a basis of the set of braid-like diagrams from a given source to a given target. This basis is given by diagrams with a minimal number of crossings and all dots placed on the bottom of the diagram, as it will be explained in the sequel.

The KLR algebras have the property to act on some endomorphism spaces of the 2-category $\dot{U}(\mathfrak{g})$: in fact, given $(I, \cdot)$ a Cartan datum, the quantum group $U(\mathfrak{g})$ associated with the Kac-Moody algebra induced by it has generators E_i and F_i for $i \in I$, and if ϵ_i is a functor that categorifies E_i, then we have an action of the KLR algebra on $\mathrm{End}(\mathcal{E}_{\underline{\epsilon}})$ where $\underline{\epsilon} = (i, j, \dots)$ is a sequence of vertices and $\mathcal{E}_{\underline{\epsilon}} = \mathcal{E}_i\mathcal{E}_j \dots$.

Thus, it is important in the process of the construction of the categorification that those algebras have explicit bases, to ensure that the Hom spaces of 2-cells in the 2-category are not either too huge nor contain too many relations that annihilate everything.

The main result of this work is to show that KLR algebras in the simply-laced case can be presented by convergent linear $(3,2)$-polygraphs, that enable us to find bases given by the set of monomials of normal forms. With a further study of these ones, we will show that there are really PBW bases in the sense of Rouquier.

The KLR algebras have the property that they can be seen as 2-categories with only one object, and with linearity on the spaces of 2-cells. In [All16], a theory of rewriting in higher dimensional categories was developed and rewriting have been applied in Affine Oriented Brauer categories, which have the same linear properties than the KLR algebras. In the framework of this article, we can see them as linear $(2,2)$-categories.

Such KLR algebras can be presented by linear $(3,2)$-polygraphs, according to [All16]; and it was proved in that paper that if a linear $(3,2)$-polygraph Σ presents such a category and is convergent, then its set of monomials in normal form Σ^m_{nf} is a basis of this category, that is to say we have

$$\Sigma^m_{nf} := \bigoplus_{p,q \in \Sigma_1} \Sigma^m_{nf}(p,q)$$

where $\Sigma^m_{nf}(p,q)$ is a basis of $\mathrm{Hom}_{\mathcal{C}}(p,q)$, the space of 2-cells between p and q, for every p and q 1-cells in the category . Then, we will present the KLR algebras with linear $(3,2)$-polygraphs and show that these latter are convergent. A further study of normal forms will then give us explicit bases, that will in fact be PBW bases in the sense of Rouquier. In higher dimension, the study of termination and confluence may be very complicated. However, for a 3-polygraph it was shown in [Gui06, GM09] that we can ensure the termination by constructing a derivation with values in a module over the category presented, as it will be recalled in the sequel. We will here explain the process and tell how it can be used in a linear framework. For the confluence, there was given in [GM09] an exhaustive list of the form of critical branchings in dimension 3. Besides, the authors studied a 3-polygraph that have similar properties as the one here, so it will enable us to shorten certain proofs.

1. PRELIMINARIES

In this first section, we introduce all the material required in this work. We recall in Section 1.1 the notion of KLR algebra and its diagrammatic version. Then, in Section 1.2 we recall the notion of linear $(3,2)$-polygraph and in Section 1.3, we explain the theorem that we will use to prove the termination of the polygraph presenting the KLR algebra.

1.1. KLR algebras

The KLR algebras generalize nil Hecke algebras and were defined by Rouquier [Rou08] or Khovanov and Lauda [KL08a]. These algebras are used to construct a categorification of some Kac-Moody algebras associated with a Cartan datum.Let Γ be a graph whose set of vertices is denoted I, and $\mathbb{K}$ any field. Let's recall the definition of Cartan datum from [Kac90].

1.1.1 Definition. A Cartan datum $(I, \cdot)$ consists of a finite set I and a bilinear form on $\mathbb{Z}[I]$, taking values in $\mathbb{Z}$ such that:

- $i \cdot i \in \{2,4,6,\dots\}$ for any $i \in I$
- $-d_{i,j} := 2\frac{i \cdot j}{i \cdot i} \in \{0,-1,-2,\dots\}$ for any $i \neq j \in I$

We say that such a Cartan datum is simply-laced if the two following conditions hold:

- For any $i \in I$, $i \cdot i = 2$
- For any $i, j \in I$, $i \cdot j \in \{0,-1\}$

We set $\mathcal{V} = \sum_{i \in I} \nu_i . i \in \mathbb{N}[I]$ an element of the free semi-group generated by I. We put $m := |\mathcal{V}| = \sum \mathcal{V}_i$. Let's also consider the set Seq$(\mathcal{V})$ which consists of all sequences of vertices of Γ with length m in which the vertex i appears exactly $\mathcal{V}_i$ times. For instance, Seq$(2i + j) = \{iij, iji, jii\}$.

1.1.2 Definition. [Rou08, Definition 3.2.1] Let $Q = (Q_{i,j})_{i,j \in I}$ a matrix with coefficients in $\mathbb{K}[u,v]$ with $Q_{i.i} = 0$ for all $i \in I$. We define a (possibly non-unitary) $\mathbb{K}$-algebra $H_{\mathcal{V}}(Q)$ by generators and relations. It is generated by elements $1_{\mathbf{i}}, x_{k,\mathbf{i}}$ for $k \in \{1,\dots,n\}$ and $\tau_{k,\mathbf{i}}$ for $k \in \{1,\dots,n\}$ and $\mathbf{i} \in$ Seq$(\mathcal{V})$. The relations are:

(1) $1_{\mathbf{i}} 1_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}} 1_{\mathbf{i}}$

(2) $\tau_{k,\mathbf{i}} = 1_{s_k(\mathbf{i})} \tau_{k,\mathbf{i}} 1_{\mathbf{i}}$

(3) $x_{k,\mathbf{i}} = 1_{\mathbf{i}} x_{k,\mathbf{i}} 1_{\mathbf{i}}$

(4) $x_{k,\mathbf{i}} x_{l,\mathbf{i}} = x_{l,\mathbf{i}} x_{k,\mathbf{i}}$

(5) $\tau_{k,s_k(\mathbf{i})} \tau_{k,\mathbf{i}} = Q_{i_k,i_{k+1}}(x_{k,\mathbf{i}}, x_{k+1,\mathbf{i}})$

(6) $\tau_{k,s_l(\mathbf{i})} \tau_{l,\mathbf{i}} = \tau_{l,s_k(\mathbf{i})} \tau_{k,\mathbf{i}}$ if $|i - j| > 1$

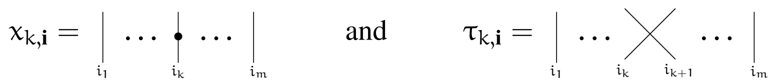
(7) $\tau_{k+1,s_k s_{k+1}(\mathbf{i})} \tau_{k,s_{k+1}(\mathbf{i})} \tau_{k+1,\mathbf{i}} - \tau_{k,s_{k+1} s_k(\mathbf{i})} \tau_{k+1,s_k(\mathbf{i})} \tau_{k,\mathbf{i}} =$
$$\begin{cases} (x_{k+2,\mathbf{i}} - x_{k,\mathbf{i}})^{-1} (Q_{i_k,i_{k+1}}(x_{k+2,\mathbf{i}}, x_{k+1,\mathbf{i}}) - Q_{i_k,i_{k+1}}(x_{k,\mathbf{i}}, x_{k+1,\mathbf{i}})) & \text{if} \quad i_k = i_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

$$(8)\quad \tau_{k,\mathbf{i}}x_{l,\mathbf{i}} - x_{s_k(l),s_k(\mathbf{i})}\tau_{k,\mathbf{i}} = \begin{cases} -1_{\mathbf{i}} & \text{if} \quad a = 1 \quad \text{and} \quad i_k = i_{k+1} \\ 1_{\mathbf{i}} & \text{if} \quad a = i+1 \quad \text{and} \quad i_k = i_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

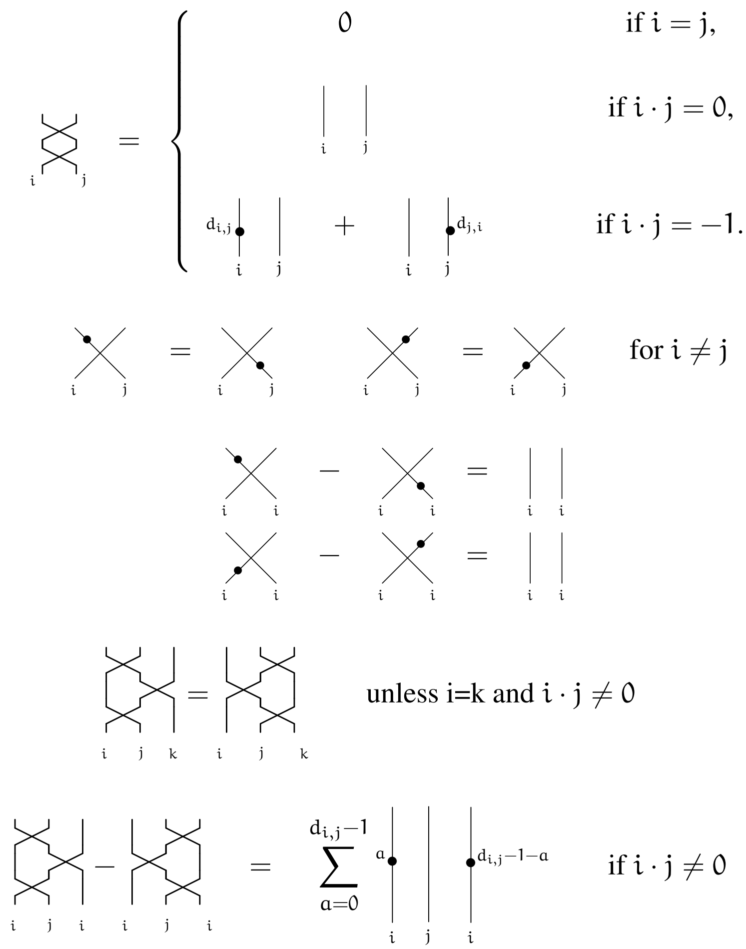
In [KL08b], Khovanov and Lauda gave a definition of a ring associated to an element $\mathcal{V} \in \mathbb{N}[I]$ which is in fact a specialization of Rouquier's algebra $H_{\mathcal{V}}(Q)$ in which we take

$$Q_{i,j}(u,v) = u^{d_{i,j}} + v^{d_{j,i}} \quad \forall \quad i,j \in I$$

Besides, they provide a diagrammatic approach for these algebras: for $\mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$, we represent the generators by the diagrams:



Then we can give a diagrammatic version of the local relations that hold in the algebra:

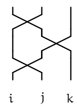
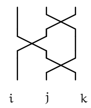


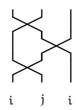
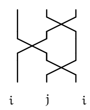
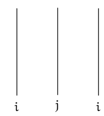
respectively for the relations (5), (8) and (7) in 1.1.2.

To simplify the computations in the sequel, we will only consider the case of a simply-laced graph: that is to say a graph without loops and multiple edges. From such a graph, we define a simply-laced Cartan datum as follows: let $\cdot$ be a bilinear form on $\mathbb{Z}[I]$ such that:

$$\begin{cases} i \cdot i = 2 & \\ i \cdot j = -1 & \text{if} \quad \text{there is an edge in} \quad \Gamma \quad \text{from i to j} \\ i \cdot j = 0 & \text{otherwise} \end{cases}$$

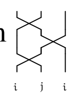
In this case, we have the coefficients $d_{i,j}$ and $d_{j,i}$ all equal to 1 when $i \cdot j = -1$ and thus the last relation becomes:

 $=$  unless $i = k$ and $i \cdot j = -1$ (1.1)

 $-$  $=$  if $i \cdot j = -1$ (1.2)

1.1.3 Definition. i) We denote by $R(\mathcal{V})$ the aforegiven algebra in the simply-laced case: we call it the simply-laced KLR Algebra.

ii) For $\mathbf{i}$ and $\mathbf{j} \in \text{Seq}(\mathcal{V})$, we define the set ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$ as the set of braid-like diagrams whose strands are labelled by vertices of Γ and such that the labels on the bottom (resp. the top) of the diagram form the sequence $\mathbf{i}$ (resp. $\mathbf{j}$).

For instance, the Yang-Baxter diagram  is an element of ${}_{iji}R(\mathcal{V})_{iji}$.

1.2. Linear $(3, 2)$-polygraphs

The general notion of linear (n, p)-categories was introduced in [All16] to deal with the fact that there is linearity on the spaces of k-cells, for $p \leq k \leq n$. The KLR algebras can be seen as 2-categories with only one 0-cell, the 1-cells given by the elements of Seq$(\mathcal{V})$ and the 2-cells given by the braid-like diagrams from a sequence to another one. As we can make sums of diagrams, we have linearity only for the spaces of 2-cells, so the KLR algebra can be seen as a $(2, 2)$-category. The linear (n, p)-polygraphs were defined inductively in [All16] to present this structure of linear (n, p)-category: more precisely, a linear (n, p)-category can be presented by linear $(n + 1, p + 1)$-polygraphs. Let us recall the notion of linear $(3, 2)$-polygraph.

1.2.1. Inductive definition. A 1-*polygraph* Σ is a graph with a set of vertices Σ_0 and a set of edges Σ_1 with *source* and *target* maps s_0, t_0: $\Sigma_1 \to \Sigma_0$. A 2-*polygraph* as a pair $\langle \Sigma_1, \Sigma_2 \rangle$ where Σ_1 is a 1-polygraph and Σ_2 is *a globular extension* of the free 1-category Σ_1^* generated by Σ_1, that is a set Σ_2 of 2-cells equipped with two maps $s_1, t_1 : \Sigma_2 \to \Sigma_1^*$ called respectively 1-source and 1-target such that the globular relations hold: $s_0 \circ s_1 = s_0 \circ t_1$ and $t_0 \circ s_1 = t_0 \circ t_1$.

Then, we construct Σ_2^l the category enriched in linear categories generated by the 2-polygraph $\Sigma = \langle \Sigma_0, \Sigma_1, \Sigma_2 \rangle$ by the category which as the same 0-cells and 1-cells then Σ and for any 2-cells p and q, $\Sigma_2^l(p, q)$ is the free vector space on $\Sigma_2^*(p, q)$ the free 2-category on Σ. *A linear* $(3, 2)$*-polygraph* is the data of $\Sigma = \langle \Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3 \rangle$ where $\langle \Sigma_0, \Sigma_1, \Sigma_2 \rangle$ is a 2-polygraph and Σ_3 is a globular extension of Σ_2^l, that is a set equipped with two *source* and *target* maps s_2, t_2: $\Sigma_3 \to \Sigma_2^l$ such that the globular relations hold: $s_0 \circ s_1 = s_0 \circ t_1$, $t_0 \circ s_1 = t_0 \circ t_1$, $s_1 \circ s_2 = s_1 \circ t_2$ and $t_1 \circ s_2 = t_1 \circ t_2$

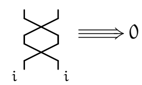
2. Convergent presentation of the KLR algebras

2.1. Definition of the linear $(3, 2)$-polygraph KLR

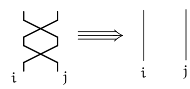
2.1.1 Definition. Let KLR be the linear $(3, 2)$-polygraph defined by:

- One 0-cell $\{*\}$
- The 1-cells are $\mathbf{i} \in \mathrm{Seq}(\mathcal{V})$ so that the generating 1-cells are $i \in I$
- The 2-cells between two 1-cells $\mathbf{i}$ and $\mathbf{j}$ are given by the diagrams in ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$
- The 3-cells are given by a choice of orientation for the relations in $R(\mathcal{V})$, that is to say

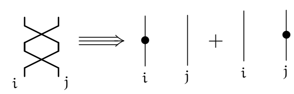
  i) For any $i \in I$,



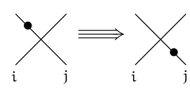
  ii) For any $i, j \in I$ such that $i \cdot j = 0$,



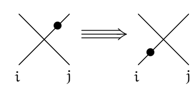
  iii) For any $i, j \in I$ such that $i \cdot j = -1$,



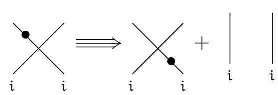
  iv) For any $i, j \in I$,



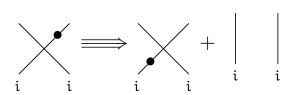
and



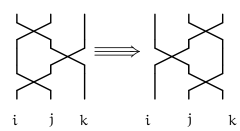
  v) For any $i \in I$,



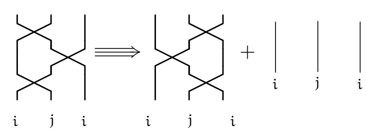
and



  vi) For any $i, j, k \in I$, and unless $i = k$ and $i \cdot j = -1$,



  vii) For any $i, j \in I$ such that $i \cdot j = -1$,



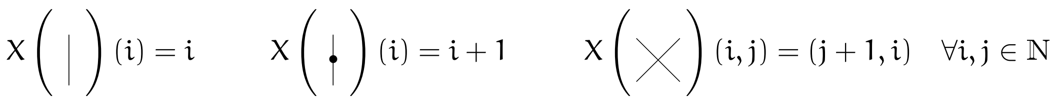
Here, we give the main theorem of this paper:

2.1.2 Theorem. *The linear* $(3, 2)$*-polygraph KLR presents the simply-laced KLR Algebra, and is terminating and confluent*

We split the proof into two parts: first, let's look at the termination and then we will study the confluence with an exhaustive study of the critical branchings of KLR.

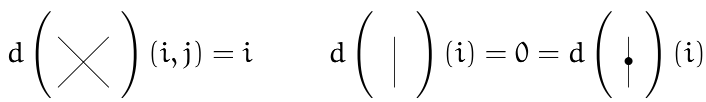
2.2. The proof of termination

Here, we will use the theorem of termination by a construction of a derivation with values in a certain module build from the 2-category, as it was explained in [GM09, Theorem 4.2.1]. This was established in a non linear case, here we use the same idea but we adapt it for a linear $(3,2)$-polygraph by requiring that we have the inequalities $X(s\alpha) \geq X(g)$, $Y(s\alpha) \geq Y(g)$ and $d(s\alpha) > d(g)$ for every $g \in \text{Supp}(t\alpha)$, where d is the derivation and $X, Y : \Sigma_2 \to$ **Ord** are 2-functors. In our case, we define a 2-functor X on generating 2-cells by:



and we set Y to be the trivial functor, so that we consider the $\mathbb{C}$-module $M_{X,*,\mathbb{Z}}$, where $\mathbb{C}$ is the category presented by Σ.

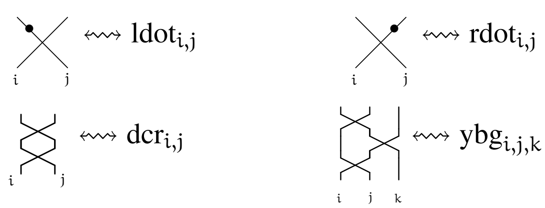
Now, we define the derivation d of KLR_2^* into $M_{X,*,\mathbb{Z}}$ given on the generating 2-cells by



We refer the reader to look at the first section of the appendix to see the inequalities that are satisfied by X and d: in fact, they satisfy the conditions of the theorem, and thus the linear $(3,2)$-polygraph KLR is terminating.

2.3. Critical branchings

To avoid drawing all the critical branchings, we will just give the exhaustive list of them by giving all the pairs of 2-cells that give rise to a critical branching.
Without looking at the vertices or the bilinear form, we have 4 different forms for the sources of 3-cells. We denote:



Without the indexations, we have 6 main families of critical branchings, which are all confluent.

A) **Crossings with two dots**

- ($\text{ldot}_{i,j}$, $\text{rdot}_{i,j}$)
- ($\text{ldot}_{i,i}$, $\text{rdot}_{i,i}$)

B) **Triple crossings**

- ($\text{dcr}_{j,i}$,$\text{dcr}_{i,j}$) for $i \cdot j = 0$ and $i \cdot j = -1$
- ($\text{dcr}_{i,i}$, $\text{dcr}_{i,i}$)

C) **Double crossings with dots**

- ($\text{ldot}_{j,i}$,$\text{dcr}_{i,j}$) for $i \cdot j = 0$ and $i \cdot j = -1$
- ($\text{rdot}_{j,i}$, $\text{dcr}_{i,j}$) for $i \cdot j = 0$ and $i \cdot j = -1$
- ($\text{ldot}_{i,i}$, $\text{dcr}_{i,i}$) ; ($\text{rdot}_{i,i}$, $\text{dcr}_{i,i}$)

D) **Double Yang-Baxter**

- ($ybg_{k,i,i}$,$ybg_{i,i,k}$) for $i \cdot k = 0$ and $i \cdot k = -1$
- ($ybg_{j,j,i}$,$ybg_{i,j,j}$) for $i \cdot j = 0$ and $i \cdot j = -1$
- ($ybg_{j,k,i}$, $ybg_{i,j,k}$) for all the values possible of $i \cdot j$, $i \cdot k$ and $j \cdot k$ (6 cases)
- ($ybg_{j,i,i}$, $ybg_{i,j,i}$) for $i \cdot j = 0$ and $i \cdot j = -1$

E) **Yang-Baxter + Crossings**

- ($ybg_{i,i,k}$, $dcr_{i,i}$)
- ($ybg_{j,i,j}$, $dcr_{i,j}$) for $i \cdot j = 0$ and $i \cdot j = -1$
- ($ybg_{j,i,k}$, $dcr_{i,j}$) for $i \cdot k = 0$ and $i \cdot k = -1$: it does not depend on the value of $i \cdot j$ or $j \cdot k$
- ($ybg_{j,i,i}$, $dcr_{i,j}$) for $i \cdot j = 0$ and $i \cdot j = -1$
- ($dcr_{j,j}$, $ybg_{i,j,j}$)
- ($dcr_{i,k}$, $ybg_{i,i,k}$) for $i \cdot k = 0$ and $i \cdot k = -1$
- ($dcr_{j,k}$, $ybg_{i,j,k}$) for $j \cdot k = 0$ and $j \cdot k = -1$: it does not depend on the value of $i \cdot k$ or $i \cdot j$
- ($dcr_{j,i}$, $ybg_{i,j,i}$) for $i \cdot j = 0$ and $i \cdot j = -1$

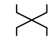
F) **Yang-Baxter + Dots**

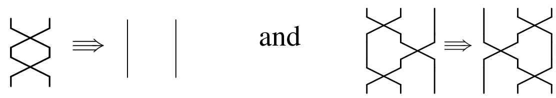
- ($ldot_{i,k}$, $ybg_{i,j,k}$) ; ($ldot_{i,j}$,$ybg_{i,j,k}$) ; ($rdot_{i,j}$, $ybg_{i,j,k}$) : it does not depend on the values of the bilinear pairing
- ($rdot_{i,k}$, $ybg_{i,i,k}$) : we can put a dot on the right strand or middle strand
- ($ldot_{j,j}$, $ybg_{i,j,i}$): we can put a dot on the left strand or middle strand
- ($ldot_{i,k}$, $ybg_{i,i,k}$) ; ($rdot_{i,j}$, $ybg_{i,j,j}$)
- ($ldot_{j,i}$, $ybg_{i,j,i}$) ; ($rdot_{j,i}$, $ybg_{i,j,i}$) ; ($rdot_{i,i}$, $ybg_{i,j,i}$) for $i \cdot j = 0$ and $i \cdot j = -1$

G) **Indexed critical branchings**
In [GM09], the authors gave an exhaustive list of the form of critical branchings for a 3-polygraph with such diagrammatic relations. If we look at their study of *the 3-polygraph of permutations*, we see that it remains to check the critical branchings with the form

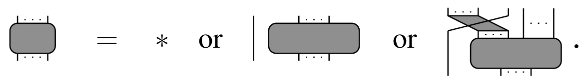
$$k$$

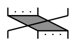
where k is a diagram that can be plugged in the Yang-Baxter-equation. This is called an *indexed critical branching (by* k*)*. It was proved in [GM09, Section 5.3] from the work of [Laf03] that to ensure the confluence, it is sufficient to check for the instances k which are in normal forms.Thus, we have now to determine which are the normal forms that we can plug in 2.1.

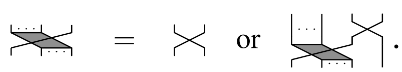
2.3.1. Study of the normal forms. In [GM09], the authors made a full study of the normal forms of the 3-polygraph of permutations Δ which has one 0-cell, one 1-cell one 2-cell 
and the following two 3-cells:



They proved using the aforegiven derivation that the set of normal forms of that polygraph is given by the set N of 2-cells of Δ^* given by the following inductive graphical scheme:



where  is itself defined inductively by

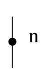
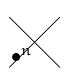


Here, the data is almost the same, except that we can add an arbitrary number of dots on each strand. For the crossings in the diagrams, they will reduce in the same way that in the polygraph of permutations, but we have to consider the dots to determine normal forms.
In fact, whenever a dot is placed on a strand and this latter intersects with another strand under the dot, a rewriting rule can be applied on it. We can thus consider a map

$$\begin{array}{rcccc} f & : & R(\mathcal{V}) & \to & \mathbb{N}^m \\ & & D & \mapsto & (c_1(D), \ldots, c_m(D)) \end{array}$$

where m is the number of inputs and outputs of the diagram D and for every $1 \leq k \leq m$, $c_k(D)$ is the number of crossing under the upper dot on the $k-th$ strand of D.
Then we immediately notice that the normal forms are the elements of N for which we have $f(D) = 0$, that is to say the elements of N in which we can place many arbitrary dots on the **bottom** of each strand.

As a consequence, the set of normal forms we can place in 2.1 are normal forms with 1 input and is an identity with many dots, or 2 inputs and is a crossing with many dots on the bottom of the leftmost strand, so as it is given by:

-  for all $n \in N$ (just the identity if $n = 0$)
-  for all $n \in \mathbb{N}$

All the diagrams that come from a right-indexation are confluent. They are all drawn in the appendix.

3. POINCARÉ-BIRKHOFF-WITT BASES

Consider the (probably non-unitary) algebra $R_n = (\mathbb{K}^{(I)}[x])^{\otimes n} = \mathbb{K}[x_1, \dots, x_n] \otimes (\mathbb{K}^{(I)})^{\otimes n}$. We denote by 1_s the idempotent corresponding to the s-th factor of $\mathbb{K}^{(I)}$ and we put $1_{\mathcal{V}} = 1_{i_1} \otimes \cdots \otimes 1_{i_m}$ for $\mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$. There is a morphism of algebras

$$\begin{array}{ccc} R_m & \to & H_{\mathcal{V}}(Q) \\ x_k 1_{\mathbf{i}} & \mapsto & x_{k,\mathbf{i}} \end{array}.$$

Let J be a set of finite sequences of elements of $\{1, \dots, m-1\}$ such that $\{s_{i_1} \dots s_{i_r}\}_{(i_1,\dots,i_r)\in J}$ is a set of minimal length representative of elements of $\mathcal{S}_n$.

In [Rou08, Theorem 3.7], Rouquier gives an algebraic definition for the KLR algebra to satisfy the PBW property: in fact, it is equivalent to the fact that the set

$$S = \{\tau_{i_1, s_{i_2} \dots s_{i_r}(\mathbf{j})} \dots \tau_{i_r,\mathbf{j}} x_{1,\mathbf{j}}^{a_1} \dots x_{m,\mathbf{j}}^{a_m}\}_{(i_1,\dots,i_r)\in J, (a_1,\dots,a_m)\in\mathbb{N}^m, \mathbf{j}\in \mathrm{Seq}(\mathcal{V})}$$

is a basis of it.

In [KL08a], Khovanov and Lauda looked at a basis for the diagrams with source $\mathbf{i}$ and target $\mathbf{j}$. First of all, they notice the fact that any diagram in $R(\mathcal{V})$ is uniquely determined by its bottom sequence and an element of $\mathcal{S}_m$. For each element $\sigma \in \mathcal{S}_m$, we can fix a minimal presentation $\tilde{\sigma}$ of σ (for the classical Coxeter presentation of the symmetric group). For $\mathbf{i}$ and $\mathbf{j} \in \mathrm{Seq}(\mathcal{V})$, they define ${}_{\mathbf{j}}S_{\mathbf{i}}$ as the subset of $\mathcal{S}_m$ containing all permutations which take $\mathbf{i}$ to $\mathbf{j}$. For each $\sigma \in {}_{\mathbf{j}}S_{\mathbf{i}}$, we fix a minimal presentation $\tilde{\sigma}$ and we consider the diagram corresponding to $\mathbf{i}$ and $\tilde{\sigma}$ in ${}_{\mathbf{j}}R_{\mathbf{i}}$, which is denoted $\hat{\sigma_i}$. We also denote ${}_{\mathbf{j}}\hat{S}_{\mathbf{i}} = \{\hat{\sigma_i}\}_{\sigma\in {}_{\mathbf{j}}S_{\mathbf{i}}}$

Thus, they showed that the set

$${}_{\mathbf{j}}B_{\mathbf{i}} = \{y x_{1,\mathbf{i}} \dots x_{m,\mathbf{i}}\}_{y\in {}_{\mathbf{j}}\hat{S}_{\mathbf{i}}, u_i\in\mathbb{N}}$$

is a free basis of the free abelian group ${}_{\mathbf{j}}R_{\mathbf{i}}$. This is equivalent to the PBW basis, since we have

$$S \simeq \bigoplus_{\mathbf{i},\mathbf{j}\in \mathrm{Seq}(\mathcal{V})} {}_{\mathbf{j}}B_{\mathbf{i}}$$

In [All16, Proposition 4.2.15], it was proved that if a linear $(3,2)$-polygraph presents a linear $(2,2)$-category, then for any one cells u and v with same 0-source and 0-target, the set of monomials of KLR in normal form, that we denote $\mathrm{KLR}_{\mathrm{nf}}^m$ is a basis of the space of 2-cells between u and v. Here, we have proved that our 3-polygraph KLR was a presentation of the simply-laced KLR algebra, so that its set of monomial in normal form $\mathrm{KLR}_{\mathrm{nf}}^m := \bigoplus_{p,q\in \mathrm{KLR}_1} \mathrm{KLR}_{\mathrm{nf}}^m(p,q)$ is a basis of the algebra seen as a linear $(2,2)$-category if we set such a basis to be a reunion of basis of each space of 2-cells. Besides, we noticed that the monomials in normal form have the following properties:

- The diagrams which are normal forms have a minimal number of crossings (that is to say strands intersecting the other), which can be interpreted as the fact that they are represented by minimal presentations of S_m
- All the normal forms with dots have their dots placed on the bottom of the diagrams: there doesn't have any crossing under a dot of a normal form.

Thus, our set of normal form is in fact given by a choice of a minimal presentation of a permutation of $\mathcal{S}_m$ such that all the elements $s_i s_{i+1} s_i$ are replaced by $s_{i+1} s_i s_{i+1}$ (to ensure that all the left Yang-Baxters are replaced by right ones) and an arbitrary number of dots placed on the bottom of each strand. Consequently, this is a basis equivalent to the ones of Rouquier or Khovanov and Lauda.
We have thus shown the second main theorem of this paper, which is in fact a direct consequence of the first one:

3.0.1 Theorem. *The simply-laced KLR algebra admits a Poincaré-Birkhoff-Witt basis.*

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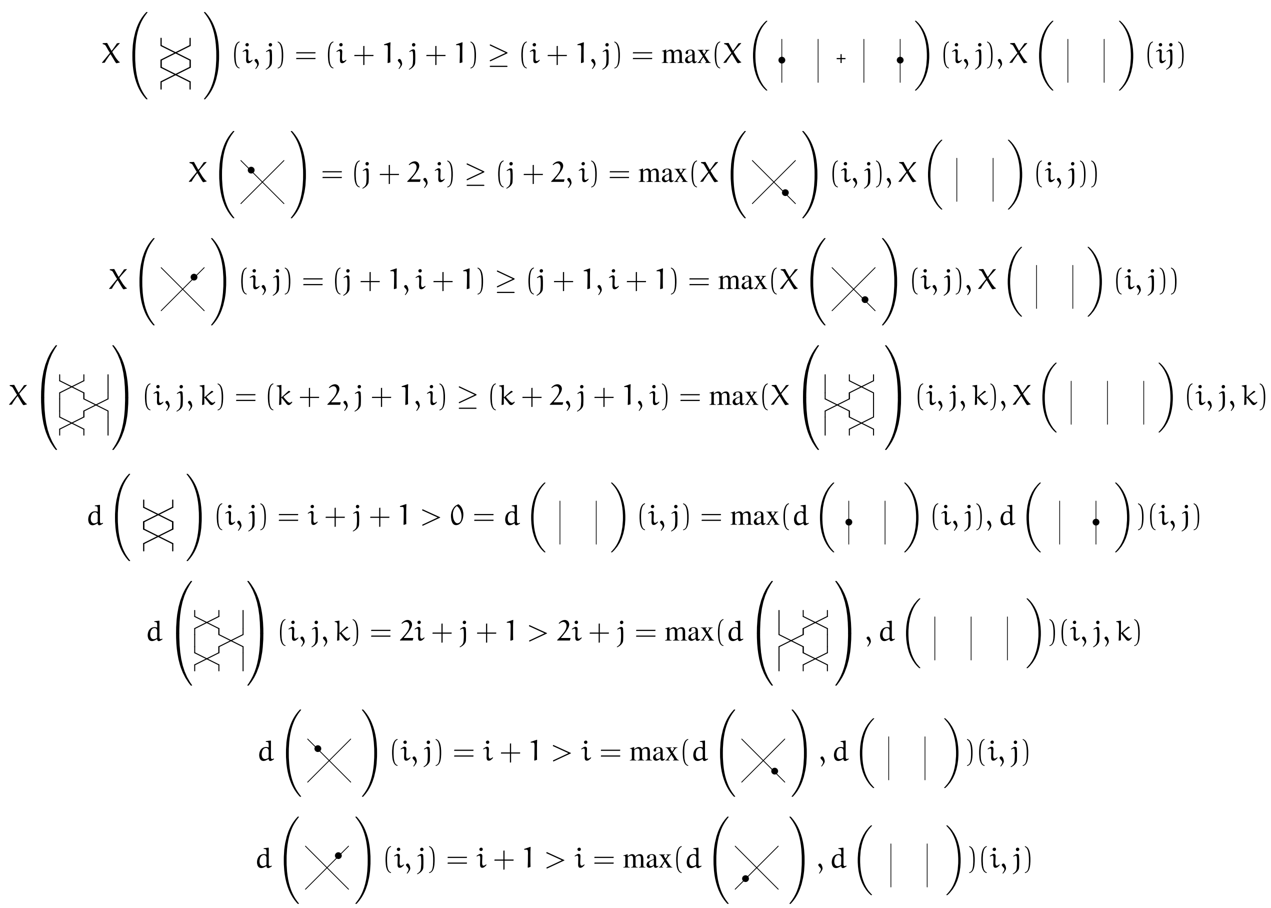
APPENDIX

3.1. The proof of termination

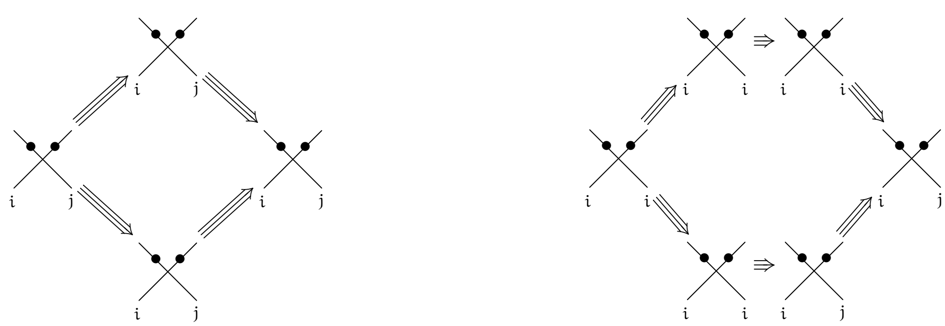
In this section, we give all the inequalities that are satisfied by the 2-functor X and the derivation d to ensure that the theorem of termination holds in our case. We have the following inequalities:



3.2. Critical branchings

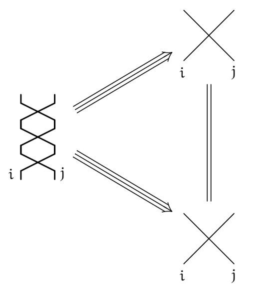
In this section, we will draw all the diagram corresponding to the given list of critical branchings for the linear $(3, 2)$-polygraph KLR.All the diagrams are drawn up by isotopy. As a consequence, when a stripe of a diagram is given by only vertical strands, we can reduce it and consider the remaining diagram.

A) **Crossings with two dots**

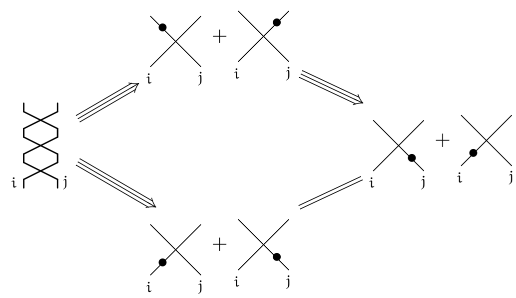


when $i \neq j$

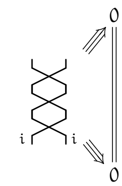
B) **Triple crossings**



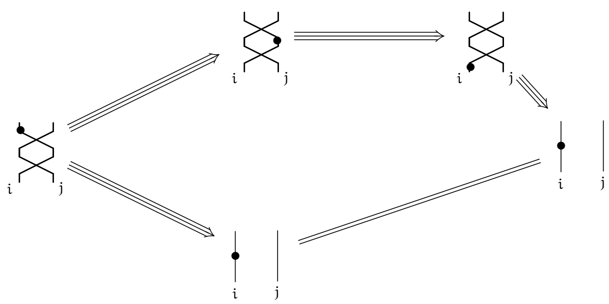
when $i \neq j$ and $i \cdot j = 0$



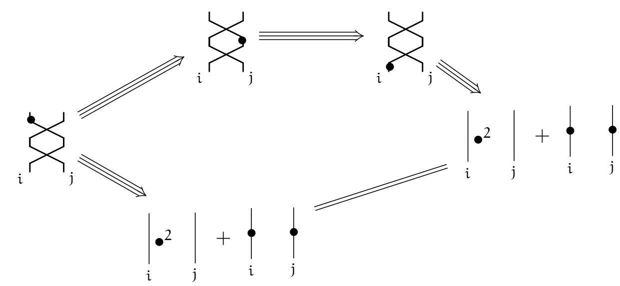
when $i \neq j$ and $i \cdot j = -1$



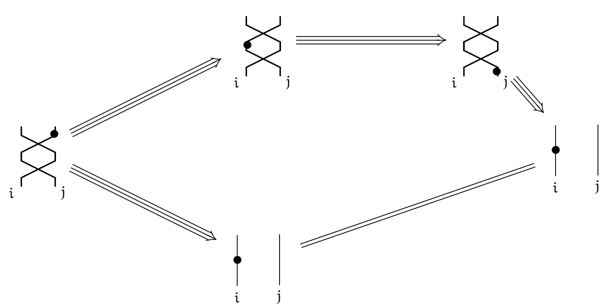
C) **Double crossings with dots**



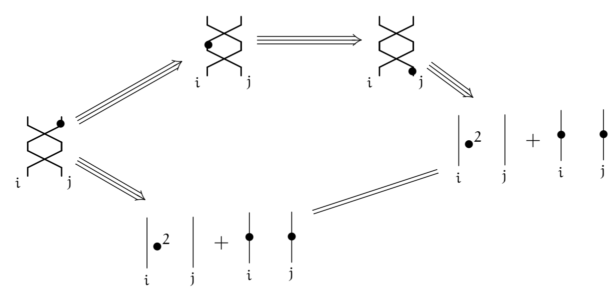
when $i \neq j$ and $i \cdot j = 0$



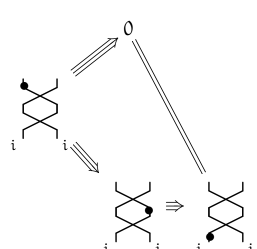
when $i \neq j$ and $i \cdot j = -1$

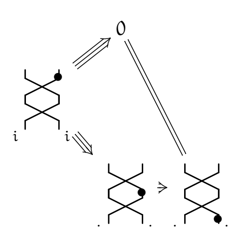


when $i \neq j$ and $i \cdot j = 0$



when $i \neq j$ and $i \cdot j = -1$

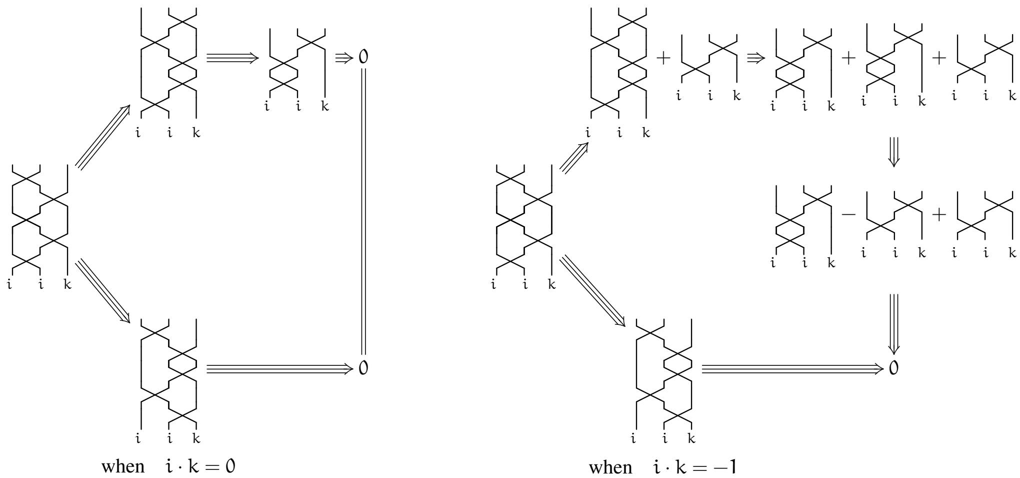




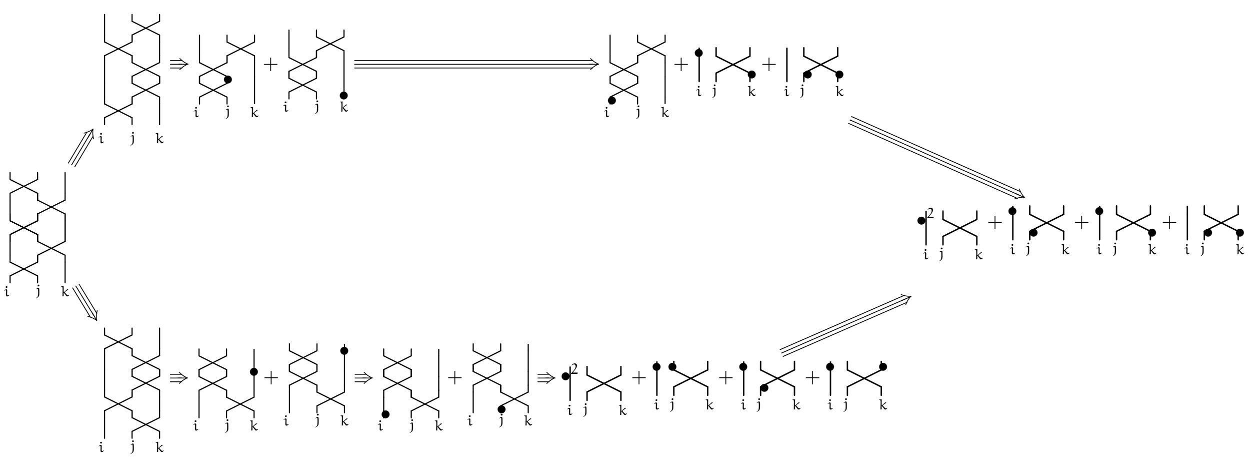
D) **Double Yang-Baxter**

The Yang-Baxter relation is made on three strands, to study all critical branchings of the superposition of two Yang-Baxters, we have to treat several case

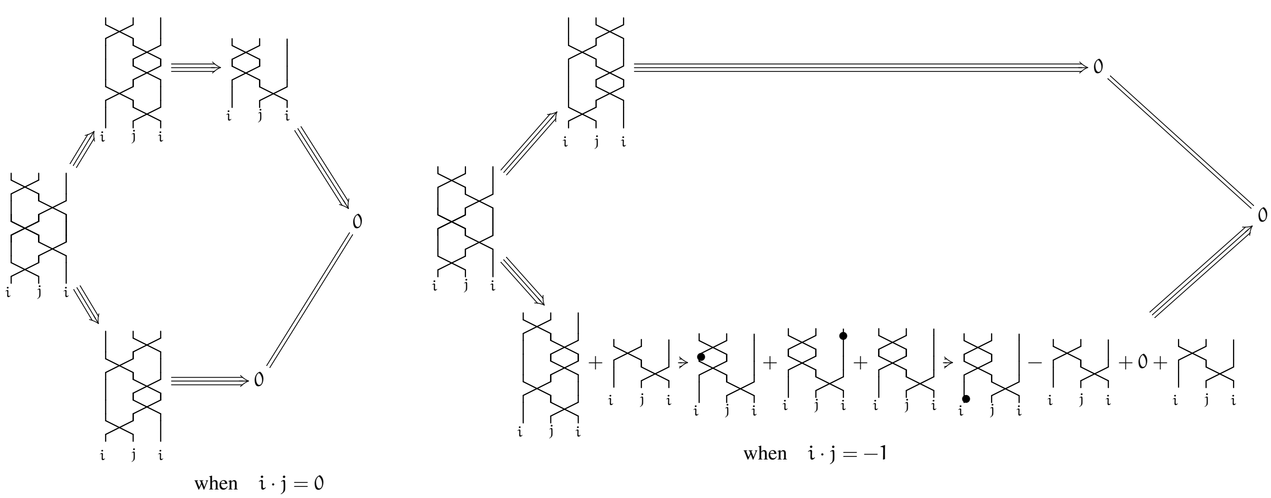
i) First of all, let's treat the case where we have two equal consecutive vertices: for instance, let's $i = j \neq k$, the other case would provide the same discussion



ii) Then, let's consider the cases where the three vertices are all distinct: we have to distinguish 6 cases according the values of $i \cdot j$, $j \cdot k$ and $i \cdot k$. Let's treat the case where $i \cdot j = i \cdot k = j \cdot k = -1$, the others will be confluent in the same way.



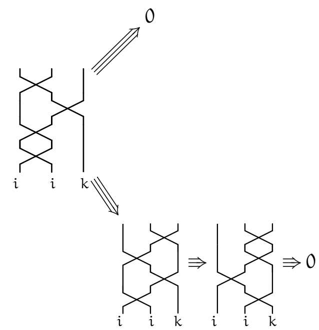
iii) Now, let's focus on the case $i = k$:



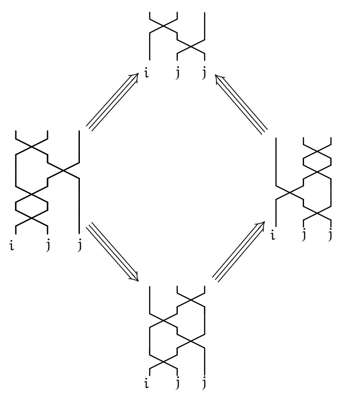
E) **Yang-Baxter + Crossings**
Here again, we have many cases to study, almost the same than in the previous section.

i) Let's begin with the case where two consecutive vertices are equal:

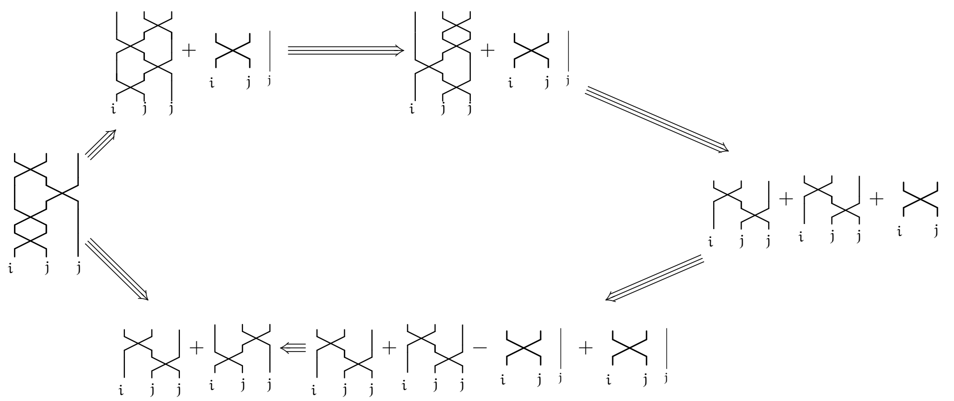
For $\mathtt{i} = \mathtt{j}$, we have



For $\mathtt{j} = \mathtt{k}$, we have

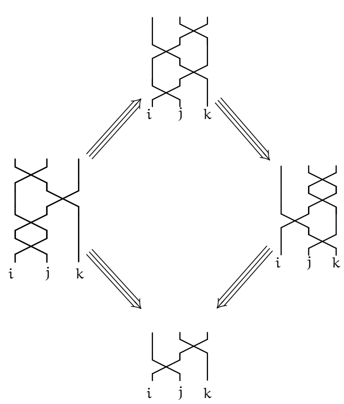


when $\mathtt{i} \cdot \mathtt{j} = 0$

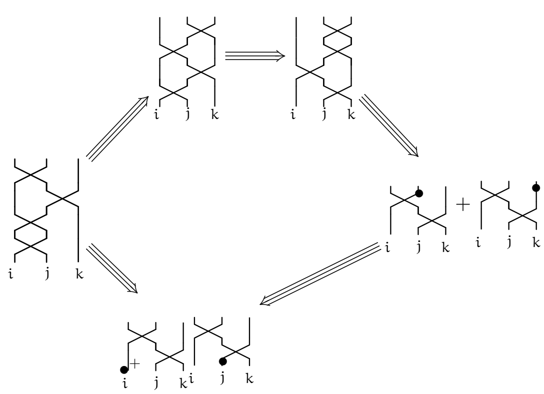


when $\mathtt{i} \cdot \mathtt{j} = -1$

ii) Now let's check the case where all the vertices are different; by computation, one gets that the critical branchings then only depend on the value of $\mathtt{i} \cdot \mathtt{k}$:

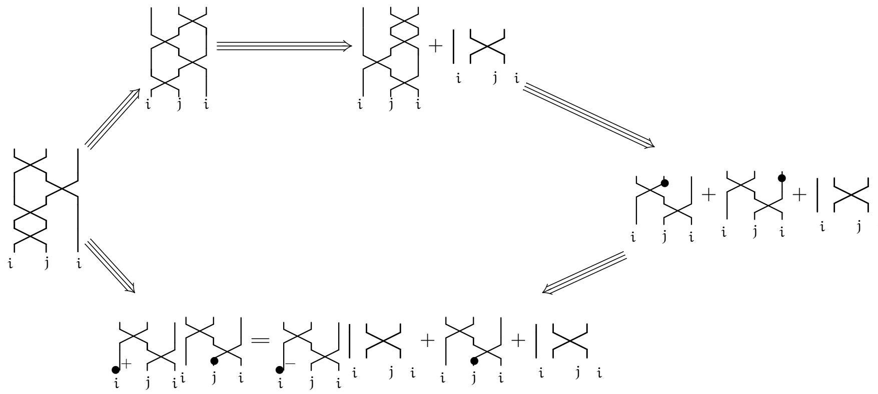


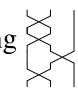
when $\mathtt{i} \cdot \mathtt{k} = 0$



when $\mathtt{i} \cdot \mathtt{k} = -1$

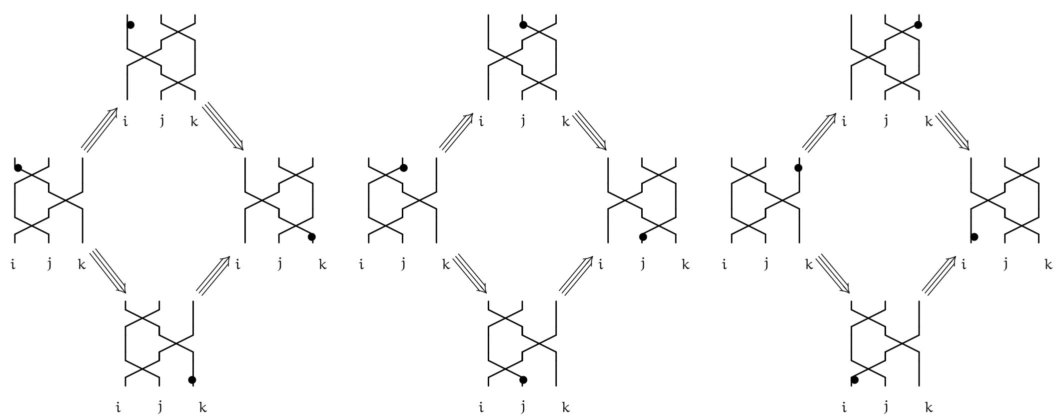
iii) We look at the case where the bottom sequence is $\mathtt{iji}$ and focus on the case $\mathtt{i} \cdot \mathtt{j} = -1$:



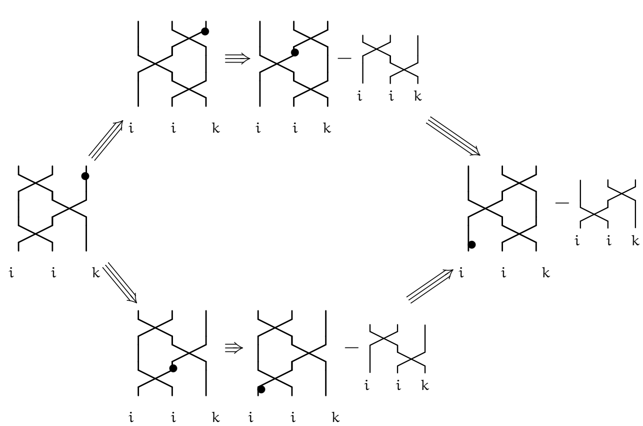
We study the confluence diagrams of all the forms of the branching  in the same way.

F) **Yang-Baxter with dots**

i) We start with the case where all the vertices are distinct: here, the diagrams we get don't depend on the values of the bilinear pairing.

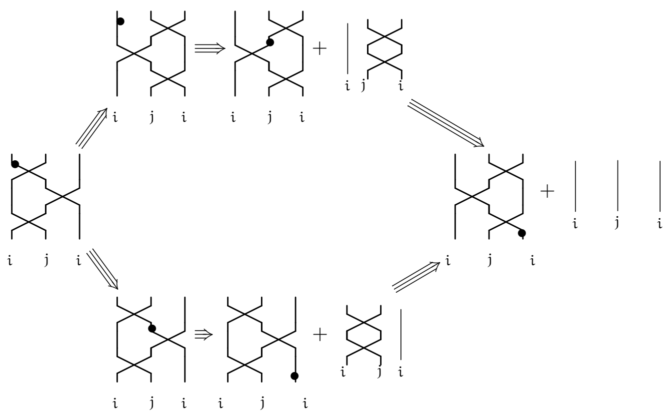


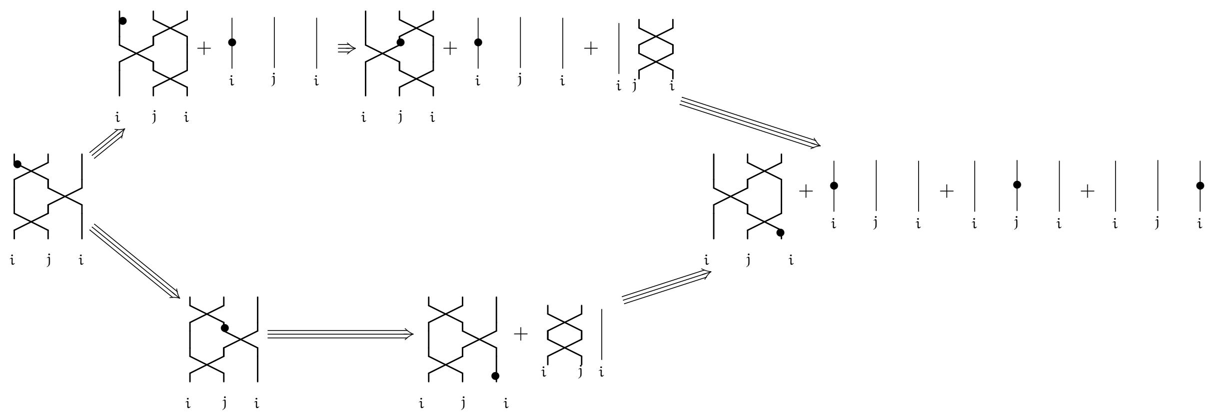
ii) Now, we look at the case where to consecutive vertices are equal: for instance, we assume that $i = j \neq k$. Let's notice that if a dot is placed on the left strand, then it will go down in the diagram without creating any additive term because there will be no crossing with two strands with the same label; so the branching is trivially confluent. For the other cases, it is the same process and we check it when there is a dot on the rightmost strand:



One may apply the same process for the case $i \neq j = k$ with a dot placed on the up of the leftmost (or middle) strand.

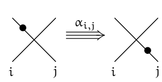
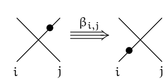
iii) Now, we focus on the case where the bottom sequence is iji: as the way to make dots go down is the same no matter where the dot is placed, we only check for a dot placed on the leftmost strand, but it would provide the same thing for the other cases.





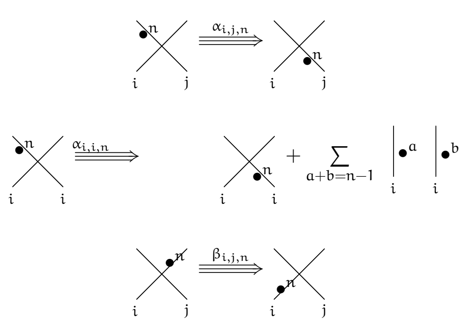
when $i \cdot j = -1$

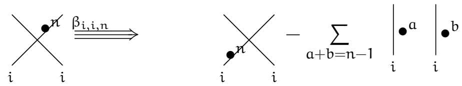
G) **The diagram of indexations**

i) For the first case, the instance for $n = 0$ was already checked in the Double Yang-Baxter family of critical branchings. Let's compute the branching in the general case: let's just look at the case $i = k$ and $i \cdot j = -1$, which is the "most complicated" case in the sense that it is the one that creates the more additive terms. First of all, if we denote $\alpha_{i,j}$ (resp. $\beta_{i,j}$) the 3-cell  (resp. ) and

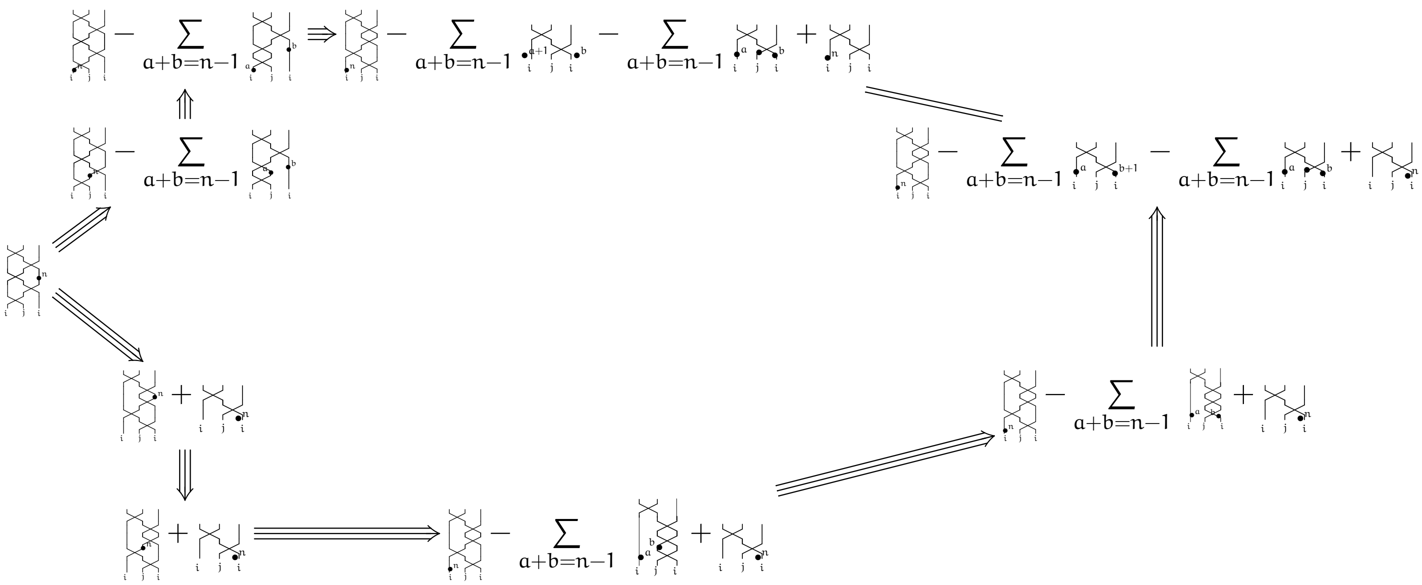
$$\alpha_{i,j,n} = \underbrace{\alpha_{i,j} \star_2 \alpha_{i,j} \cdots \star_2 \alpha_{i,j}}_{n \quad \text{times}} \qquad \text{(resp. } \beta_{i,j,n} = \underbrace{\beta_{i,j} \star_2 \beta_{i,j} \cdots \star_2 \beta_{i,j}}_{n \quad \text{times}}\text{)}$$

we have the following 3-cells in Σ:

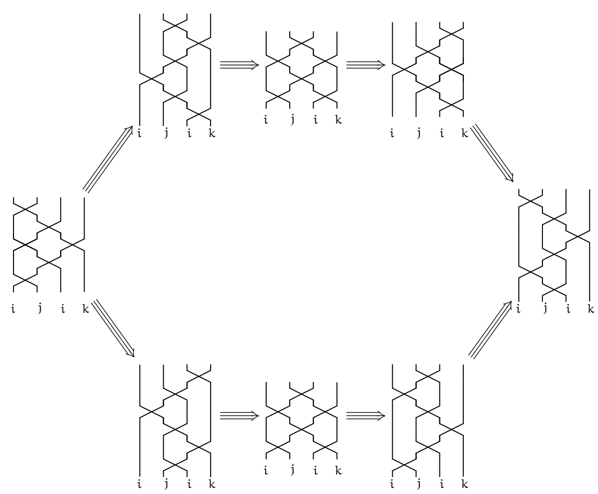


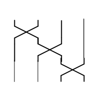


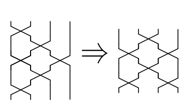
Thus, we have:

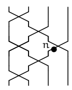


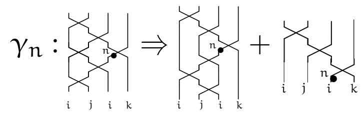
ii) For the second indexation, one remarks by computing that the fourth vertex of the sequences doesn't matter in the reduction rule we can apply on a diagram. For the sequel, we will thus consider the case where the bottom sequence is $ijik$ with $i \cdot j = -1$.
Let's look at this indexation for $n = 0$:



This diagram was given in [GM09] for the indexation of  in the double Yang-Baxter diagram. When $i \cdot j = -1$, it is the same diagram except that it creates an extra term  in both sides of the critical branching. Let's look at the general case for $n > 0$. The bottom line of **??** defines a 2-cell

$\gamma :$ 

. As we started reducing only the bottom part on the diagram, we can apply the same reduction by putting many dots as in  such that the dots will never be placed in the source of any reduction we apply on the diagrams. This enables us to define, for every $n \in \mathbb{N}$, a 2-cell



Then we have: